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Scheduling of unit-length jobs with cubic incompatibility graphs on three uniform machines[☆]

Hanna Furmańczyk[a,*], Marek Kubale[b]

[a] Institute of Informatics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

[b] Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

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ABSTRACT

In the paper we consider the problem of scheduling n identical jobs on 3 uniform machines with speeds s_1, s_2, and s_3 to minimize the schedule length. We assume that jobs are subjected to some kind of mutual exclusion constraints, modeled by a cubic incompatibility graph. We show that if the graph is 2-chromatic then the problem can be solved in $O(n^2)$ time. If the graph is 3-chromatic, the problem becomes NP-hard even if $s_1 > s_2 = s_3$. However, in this case there exists a 10/7-approximation algorithm running in $O(n^3)$ time. Moreover, this algorithm solves the problem almost surely to optimality if $3s_1/4 \leq s_2 = s_3$.

1. Introduction

Imagine you have to arrange a dinner for, say 30, people and you have at your disposal 3 round tables with different numbers of seats (not greater than 15). You know that each of your guests is in bad relations with exactly 3 other people. Your task is to assign the people to the tables in such a way that no two of them being in bad relations seat at the same table. In the paper we show how to solve this and related problems.

Our problem can be expressed as the following scheduling problem. Suppose we have n identical jobs $j_1, \ldots, j_n$, so we assume that they all have unit execution times, in symbols $p_i = 1$, to be processed on three non-identical machines M_1, M_2, and M_3. These machines run at different speeds s_1, s_2, and s_3, respectively. However, they are *uniform* in the sense that if a job is executed on machine M_i, it takes $1/s_i$ time units to be completed. It refers to the situation where the machines are of different generations, e.g. old and slow, new and fast, etc.

Our scheduling model would be trivial if all the jobs were compatible. Therefore we assume that some pairs of jobs cannot be processed on the same machine due to some technological constraints. More precisely, we assume that each job is in conflict with exactly three other jobs. Thus the underlying *incompatibility graph G* whose vertices are jobs and edges correspond to pairs of jobs being in conflict is cubic. For example, all graphs in our figures are cubic. The number of jobs n must be even, since the sum of degrees of all vertices in G, i.e. $3n$, must be even. A load L_i on machine M_i requires the

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* Corresponding author.
E-mail addresses: hanna@inf.ug.edu.pl (H. Furmańczyk), kubale@eti.pg.gda.pl (M. Kubale).

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processing time $P(L_i) = |L_i|/s_i$, and all the jobs are ready for processing at the same time. By definition, each load forms an independent set (color) in G. Therefore, in what follows we will be using the terms job/vertex and color/independent set interchangeably. Since all tasks have to be executed, the problem is to find a 3-coloring, i.e. a decomposition of G into 3 independent sets I_1, I_2, and I_3 such that the schedule length $C_{\max} = \max\{P(I_i) : i = 1, 2, 3\}$ is minimized, in symbols $Q3|p_i = 1, G = cubic|C_{\max}$.

In this paper we assume three machines for the following reason. If there is only one machine then there is no solution. If there are two machines, the problem becomes trivial because it is solvable only if G is bipartite and it has only one solution since there is just one decomposition of G into sets I_1 and I_2, each of size $n/2$. If, however, there are three machines and G is 3-chromatic, our problem becomes NP-hard. Again, if G is 4-chromatic (and $m = 3$), there is no solution.

There are several papers devoted to chromatic scheduling in the presence of mutual exclusion constraints. Boudhar in [1, 2] studied the problem of batch scheduling with complements of bipartite and split graphs, respectively. Finke et al. [8] considered the problem with complements of interval graphs. Other models of batch scheduling with incompatibility constraints were studied in [5,6]. Our problem can also be viewed as a particular variant of scheduling with conflicts [7]. In all the papers the authors assumed identical parallel machines. However, to the best of our knowledge little work has been done on scheduling problems with uniform machines involved (cf. Li and Zhang [12]).

The rest of this paper is split into two parts depending on the chromaticity of cubic graphs. In Section 2 we consider 2-chromatic graphs. In particular, we give an $O(n^2)$-time algorithm for optimal scheduling of such graphs. Section 3 is devoted to 3-chromatic graphs. In particular, we give an NP-hardness proof and an approximation algorithm with good performance guarantee. Our algorithm runs in $O(n^3)$ time to produce a solution of value less than 10/7 times optimal, provided that $s_1 > s_2 = s_3$. Moreover, this algorithm solves the problem almost surely to optimality if $3s_1/4 \le s_2 = s_3$. Finally, we discuss possible extensions of our model to arbitrary job lengths, to disconnected graphs, and to more than three machines.

2. Scheduling of 2-chromatic graphs

We begin with introducing some basic notions concerning graph coloring. A graph $G = (V, E)$ is said to be *equitably k-colorable* if and only if its vertex set can be partitioned into independent sets $V_1, \ldots, V_k \subset V$ such that $\||V_i| - |V_j|\| \le 1$ for all $i, j = 1, \ldots, k$. The smallest k for which G admits such a coloring is called the *equitable chromatic number* of G and denoted $\chi_=(G)$. Graph G has a *semi-equitable k-coloring*, if there exists a partition of its vertices into independent sets $V_1, \ldots, V_k \subset V$ such that one of these subsets, say V_i, is of size $\notin \{\lfloor n/k \rfloor, \lceil n/k \rceil\}$, and the remaining subgraph $G - V_i$ is equitably $(k-1)$-colorable. In the following we will say that graph G has $(V_1, \ldots, V_k)$-coloring to express explicitly a partition of V into k independent sets. If, however, only cardinalities of color classes are important, we will use the notation $[|V_1|, \ldots, |V_k|]$.

Let us recall some basic facts concerning colorability of cubic graphs. It is well known from Brooks theorem [3] that for any cubic graph $G \ne K_4$ we have $\chi(G) \le 3$, where $\chi(G)$ is the classical chromatic number of G and K_4 is the complete graph on four vertices. On the other hand, Chen et al. [4] proved that every 3-chromatic cubic graph can be equitably colored without introducing a new color. Moreover, since a connected cubic graph G with $\chi(G) = 2$ is a bipartite graph with partition sets of equal size, we have the equivalence of the classical and equitable chromatic numbers for 2-chromatic cubic graphs. Since the only cubic graph for which the chromatic number is equal to 4 is K_4, we have

$$2 \le \chi_=(G) = \chi(G) \le 4 \tag{1}$$

for any cubic graph. Moreover, from (1) it follows that for any cubic graph $G \ne K_4$, we have

$$n/3 \le \alpha(G) \le n/2 \tag{2}$$

where $\alpha(G)$ is the independence number of G. Note that the upper bound is tight only if G is bipartite.

Let $\mathcal{Q}_k$ denote the class of connected k-chromatic cubic graphs and let $\mathcal{Q}_k(n) \subset \mathcal{Q}_k$ stand for the subclass of cubic graphs on n vertices, $k = 2, 3, 4$. Clearly, $\mathcal{Q}_4 = \{K_4\}$. In what follows we will call the graphs belonging to $\mathcal{Q}_2$ *bicubic*, and the graphs belonging to $\mathcal{Q}_3$- *tricubic*.

As mentioned, if G is bicubic then any 2-coloring of it is equitable and there may be no equitable 3-coloring (cf. $K_{3,3}$ shown in Fig. 1). On the other hand, all graphs in $\mathcal{Q}_2(n)$ have a semi-equitable 3-coloring of type $[n/2, \lceil n/4 \rceil, \lfloor n/4 \rfloor]$. Moreover, they are easy colorable in linear time while traversing them in a depth-first search (DFS) manner.

Let s_i be the speed of machine M_i for $i = 1, 2, 3$, and let $s = s_1 + s_2 + s_3$. Without loss of generality we assume that $s_1 \ge s_2 \ge s_3$. If there are just 6 jobs to schedule then the incompatibility graph $G = K_{3,3}$ and there is only one decomposition of it into 3 independent sets shown in Fig. 1(a), as well as there is only one decomposition of G into 2 independent sets shown in Fig. 1(b), of course up to isomorphism. The length of minimal schedule is $\min\{\max\{3/s_1, 2/s_2, 1/s_3\}, 3/s_2\}$. Therefore, we assume that our graphs have at least 8 vertices.

By an *ideal schedule* we mean a schedule in which:

(i) machine M_1 performs as many jobs as possible and M_2 and M_3 finish at the same time, if $s_1 \ge s_2 + s_3$, or
(ii) machines M_1, M_2, and M_3 all finish at the same time, if $s_1 < s_2 + s_3$.

An example of ideal schedule is shown in Fig. 2(a).