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On the double Roman domination in graphs

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ABSTRACT

A double Roman dominating function (DRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then vertex v has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then vertex v must have at least one neighbor w with $f(w) \geq 2$. The weight of a DRDF is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The double Roman domination number $\gamma_{dR}(G)$ is the minimum weight of a DRDF on G. First we show that the decision problem associated with $\gamma_{dR}(G)$ is NP-complete for bipartite and chordal graphs. Then we present some sharp bounds on the double Roman domination number which partially answer an open question posed by Beeler et al. (2016) in their introductory paper on double Roman domination. Moreover, a characterization of graphs G with small $\gamma_{dR}(G)$ is provided.

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by n. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d_G(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* of G is a vertex of degree one, while a *support vertex* of G is a vertex adjacent to a leaf.

We write P_n for the *path* of order n, C_n for the *cycle* of length n and $\overline{K_n}$ for the graph with n vertices and no edges. A *tree* is an acyclic connected graph. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. A connected graph G is *unicyclic* if it has a unique cycle. A connected subgraph B of G is a *block*, if B has no cut vertex and every subgraph $B' \subseteq G$ with $B \subsetneq B'$ has at least one cut vertex. A graph G is a *block graph*, if every block is complete. For terminology and notation on graph theory not given here, the reader is referred to [16].

A set $S \subseteq V$ in a graph G is called a *dominating set* if every vertex of G is either in S or adjacent to a vertex of S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A subset $S \subseteq V$ is a 2-*dominating set* if every vertex of $V - S$ has at least two neighbors in S. The 2-*domination number* $\gamma_2(G)$ equals the minimum cardinality of a 2-dominating set in G.

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For a graph G and a positive integer k, let $f : V(G) \to \{0, 1, 2, \ldots, k\}$ be a function, and let $(V_0, V_1, V_2, \ldots, V_k)$ be the ordered partition of $V = V(G)$ induced by f, where $V_i = \{v \in V : f(v) = i\}$ for $i \in \{0, 1, \ldots, k\}$. There is a 1–1 correspondence between the functions $f : V \to \{0, 1, 2, \ldots, k\}$ and the ordered partitions $(V_0, V_1, V_2, \ldots, V_k)$ of V, so we will write $f = (V_0, V_1, V_2, \ldots, V_k)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. in [7] and was inspired by the work of ReVelle and Rosing [13], and Stewart [14]. Since 2004, a hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [10], maximal Roman domination [2], mixed Roman domination [3], and very recently Roman {2}-domination [5] and double Roman domination [4], further studied in [1].

A *Roman* {2}*-dominating function* is a function $f : V \to \{0, 1, 2\}$ with the property that for every vertex $v \in V$ with $f(v) = 0$, $f(N(v)) \geq 2$, that is, there is a vertex $u \in N(v)$, with $f(u) = 2$, or there are two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. The weight of a Roman {2}-dominating function is the value $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman {2}-dominating function is called the *Roman* {2}*-domination number* and denoted by $\gamma_{\{R2\}}(G)$. It is worth mentioning that Roman {2}-domination was called in [12] the Italian domination.

As defined in [4], a function $f : V \to \{0, 1, 2, 3\}$ is a *double Roman dominating function* (DRDF) on a graph G if the following conditions hold.

(i) If $f(v) = 0$, then v must have one neighbor in V_3 or at least two neighbors in V_2.

(ii) If $f(v) = 1$, then v must have at least one neighbor in $V_2 \cup V_3$.

The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on G, and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G.

In this paper, we first show that the decision problem associated with $\gamma_{dR}(G)$ is NP-complete for bipartite and chordal graphs. Then we present some sharp bounds on the double Roman domination number which partially answer an open question posed by Beeler et al. in [4]. Moreover, we give the exact values of the double Roman domination number for paths and cycles, and we provide a characterization of graphs G with small $\gamma_{dR}(G)$.

2. Preliminary results

We make use of the following results in this paper.

Theorem A (*Beeler et al. [4]*). *For any double Roman dominating function f', there exists a double Roman dominating function f of no greater weight than f' for which no vertex is assigned the value* 1.

As defined in [11], for positive integers r and s, let $F_{r,s}$ be the tree obtained from a double star $S_{r,s}$ by subdividing every edge exactly once. Let $\mathcal{F}$ be the family of all such trees $F_{r,s}$; that is, $\mathcal{F} = \{F_{r,s} \mid r, s \geq 1\}$. Also, let $\mathcal{T}$ be the family of trees $T_{k,j}$ of order $k \geq 2$, where $k \geq 2j + 1$ and $j \geq 0$, obtained from a star by subdividing j edges exactly once. We call the pivot vertex of a $T_{k,j}$ the vertex that was the center of the star we started with, prior to subdividing edges.

Theorem B (*Klostermeyer and MacGillivray [12]*). *If T is a non-trivial tree, then $\gamma_{\{R2\}}(T) \geq \gamma(T) + 1$.*

Theorem C (*Henning and Klostermeyer [11]*). *Let T be a non-trivial tree. Then $\gamma_{\{R2\}}(T) = \gamma(T) + 1$ if and only if $T \in \mathcal{F} \cup \mathcal{T}$.*

For the rest of this section, we give the exact values of the double Roman domination number for paths and cycles. Then we characterize all graphs with small double Roman domination number. Recall that the join $G \vee H$ of two graphs G and H is a graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H.

Proposition 1. *for $n \geq 1$, $\gamma_{dR}(P_n) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 3 \\ n+1 & \text{if} \quad n \equiv 1 \text{ or } 2 \pmod 3. \end{cases}$*

Proof. Let $P := v_1 v_2 \ldots v_n$ be a path of order n. Define $h : V(P_n) \to \{0, 1, 2, 3\}$ by $h(v_{3i-1}) = 3$ for $1 \leq i \leq \frac{n}{3}$ and $h(x) = 0$ otherwise if $n \equiv 0 \pmod 3$, by $h(v_n) = 2$, $h(v_{3i-1}) = 3$ for $1 \leq i \leq \frac{n-1}{3}$ and $h(x) = 0$ otherwise when $n \equiv 1 \pmod 3$, and by $h(v_n) = 3$, $h(v_{3i-1}) = 3$ for $1 \leq i \leq \frac{n-2}{3}$ and $h(x) = 0$ otherwise if $n \equiv 2 \pmod 3$. It is easy to verify that h is a DRDF of P_n of weight n if $n \equiv 0 \pmod 3$ and of weight $n + 1$ if $n \not\equiv 0 \pmod 3$. Thus

$$\gamma_{dR}(P_n) \leq \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 3 \\ n+1 & \text{if} \quad n \equiv 1 \text{ or } 2 \pmod 3. \end{cases}$$

Now we prove the inverse inequality. The proof is by induction on n. The statement holds for all paths of order $n \leq 4$. For the inductive hypothesis, let $n \geq 5$ and suppose that for every path of order less than n the result is true. Assume $f = (V_0, V_1, V_2, V_3)$ is a γ_{dR}-function of P_n so that $V_1 = \emptyset$. First let $f(v_n) = 0$. Then we have $f(v_{n-1}) = 3$. Define