Discrete Applied Mathematics 232 (2017) 50–63

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On oriented cliques with respect to push operation

Julien Bensmail [a], Soumen Nandi [b], Sagnik Sen [c,*]

[a] Université Côte d'Azur, Inria, CNRS, I3S, France
[b] Indian Statistical Institute, Kolkata, India
[c] Ramakrishana Mission Vivekananda University, Kolkata, India

ARTICLE INFO

Article history:
Received 17 November 2015
Received in revised form 25 July 2017
Accepted 27 July 2017
Available online 24 August 2017

Keywords:
Oriented graphs
Oriented cliques
Push operation
Planar graphs

ABSTRACT

An oriented graph is a directed graph without any directed cycle of length at most 2. An oriented clique is an oriented graph whose non-adjacent vertices are connected by a directed 2-path. To push a vertex v of an oriented graph is to change the orientations of all the arcs incident to v. A push clique is an oriented clique that remains an oriented clique even if one pushes any set of vertices of it. We show that it is NP-complete to decide if an undirected graph is the underlying graph of a push clique or not. We also prove that a planar push clique can have at most 8 vertices and provide an exhaustive list of planar push cliques.

1. Introduction and main results

An *oriented graph* $\overrightarrow{G}$ is a directed graph with vertex set $V(\overrightarrow{G})$ and arc set $A(\overrightarrow{G})$ having no directed cycle of length 1 or 2. We denote by G the underlying graph of $\overrightarrow{G}$. An *orientation* of G is an oriented graph obtained by assigning each edge uv of G one of the two possible orientations, namely, $\overrightarrow{uv}$ or $\overrightarrow{vu}$.

An *oriented k-coloring* of an oriented graph $\overrightarrow{G}$ is a mapping f from the vertex set $V(\overrightarrow{G})$ to a set of k colors such that (i) $f(u) \neq f(v)$ whenever u and v are adjacent and, (ii) if $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in $\overrightarrow{G}$, then $f(u) = f(x)$ implies $f(v) \neq f(w)$. The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the smallest integer k for which $\overrightarrow{G}$ admits an oriented k-coloring. Oriented coloring is a well studied topic (see the latest survey [16] for details).

To *push* a vertex v of a directed graph $\overrightarrow{G}$ is to change the orientations of all the arcs (that is, to replace an arc $\overrightarrow{xy}$ by $\overrightarrow{yx}$) incident to v. The study of push operation was introduced by Mosesian [9] and studied in [2,4,7,8,11–13].

Two orientations $\overrightarrow{G}$ and $\overrightarrow{G}'$ of G are in a *push relation* if one can obtain $\overrightarrow{G}'$ by pushing a set of vertices of $\overrightarrow{G}$. The *pushable chromatic number* $\chi_p(\overrightarrow{G})$, introduced by Klostermeyer and MacGillivray [6], of an oriented graph $\overrightarrow{G}$ is the minimum oriented chromatic number taken over all oriented graphs that are in push relation with $\overrightarrow{G}$. Following the work of Klostermeyer and MacGillivray [6], the pushable chromatic number of planar graphs was studied by Sen [15].

An oriented clique, introduced by Klostermeyer and MacGillivray [5], is an oriented graph $\overrightarrow{C}$ with $\chi_o(\overrightarrow{C}) = |V(\overrightarrow{C})|$. In fact, an oriented clique is characterized as an oriented graph in which each pair of non-adjacent vertices is connected by a directed 2-path. Due to this characterization oriented cliques can be viewed as natural objects. Moreover, they play a significant role in studying oriented coloring as pointed out in [10]. An undirected simple graph is called an *underlying oriented clique* if it is the underlying graph of an oriented clique.

* Corresponding author.
E-mail addresses: julien.bensmail.phd@gmail.com (J. Bensmail), soumen2004@gmail.com (S. Nandi), sen007isi@gmail.com (S. Sen).

http://dx.doi.org/10.1016/j.dam.2017.07.037
0166-218X/

Let $\overrightarrow{C}$ be an oriented clique such that each oriented graph in a push relation with $\overrightarrow{C}$ is also an oriented clique. We are interested in such oriented cliques. Observe that $\overrightarrow{C}$ is such an oriented clique if and only if $\chi_p(\overrightarrow{C}) = |V(\overrightarrow{C})|$. Thus we define the following notion: an oriented graph $\overrightarrow{C}$ is a *push clique* if $\chi_p(\overrightarrow{C}) = |V(\overrightarrow{C})|$. Also an undirected simple graph is an *underlying push clique* if it is the underlying graph of a push clique.

Given an undirected simple graph it is NP-hard to determine if it is an underlying oriented clique [1]. We prove an analogous result for underlying push cliques.

Theorem 1.1. *It is NP-complete to decide whether a given graph is an underlying push clique.*

Oriented cliques of planar and outerplanar graphs are studied in detail, see [10]. Settling a conjecture of Klostermeyer and MacGillivray [5], it is proved in [10] that a planar oriented clique can have at most 15 vertices. Note that there exists a planar oriented clique on 15 vertices which implies that the above mentioned bound is tight. Here, we exhibit all planar push cliques, in particular proving that any such clique has at most 8 vertices.

Theorem 1.2. *A planar push clique can have at most 8 vertices. Moreover, there exists a planar push clique on 8 vertices.*

Klostermeyer and MacGillivray showed that an outerplanar oriented clique can have at most 7 vertices and any outerplanar oriented clique must have a particular oriented clique as a spanning subgraph [5]. Later this result was extended by providing an explicit list of eleven outerplanar graphs and proving that any outerplanar underlying oriented clique must have one of the eleven outerplanar graphs as a spanning subgraph [10]. In the same article the following question was asked: "Characterize the set L of graphs such that any planar graph is an underlying oriented clique if and only if it contains one of the graphs from L as a spanning subgraph." Here we answer an analogous version of this question for planar underlying push cliques.

Theorem 1.3. *An undirected planar graph is an underlying push clique if and only if it contains an underlying graph of one of the 16 planar graphs depicted in Fig. 1 as a spanning subgraph.*

In Section 2 we introduce some basic definitions and notations. The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 3, 4 and 5, respectively. Theorem 1.2 was published in EuroComb 2013 [14].

2. Preliminaries

For an (oriented) graph G every parameter we introduce below is denoted using G as a subscript. In order to simplify the notations, this subscript will be dropped whenever there is no chance of confusion.

The set of all adjacent vertices of a vertex v of an (oriented) graph G is called its set of *neighbors* and is denoted by $N_G(v)$. If there is an arc $\overrightarrow{uv}$, then u is an *in-neighbor* of v and v is an *out-neighbor* of u. The set of all in-neighbors and the set of all out-neighbors of v are denoted by $N^-_G(v)$ and $N^+_G(v)$, respectively. The *degree* of a vertex v of an (oriented) graph G, denoted by $d_G(v)$, is the number of neighbors of v in G. Naturally, the *in-degree* (resp. *out-degree*) of a vertex v of an oriented graph G, denoted by $d^-_G(v)$ (resp. $d^+_G(v)$), is the number of in-neighbors (resp. out-neighbors) of v in G. The *order* $|V(G)|$ of an (oriented) graph G is the cardinality of its set of vertices $V(G)$.

Two vertices u and v of an oriented graph *agree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+, -\}$. Two vertices u and v of an oriented graph *disagree* on a third vertex $w \in N(u) \cap N(v)$ if u and v do not agree on w.

A *k-cycle* is an undirected cycle having k vertices. Let $\overrightarrow{C}_4$ be an oriented 4-cycle with arcs ab, bc, cd, ad. A *special 4-cycle* is an oriented 4-cycle isomorphic to $\overrightarrow{C}_4$. Note that all the oriented graphs which are in push relation with a special 4-cycle are isomorphic to it. Notice that the non-adjacent vertices of a special 4-cycle always get different colors as they are always connected with a 2-dipath, no matter which vertex of the graph you push. This is, in fact, a necessary and sufficient condition for two non-adjacent vertices to receive two distinct colors under any oriented coloring with respect to the push relation.

Lemma 2.1. *An oriented graph $\overrightarrow{G}$ is a push clique if and only if any two non-adjacent vertices of $\overrightarrow{G}$ are part of a special 4-cycle.*

Due to Lemma 2.1 we know that a push clique is an oriented graph with each pair of non-adjacent vertices agreeing on at least one vertex and disagreeing on at least one vertex.

Corollary 2.2. *Each pair of non-adjacent vertices of a push clique must have at least two common neighbors.*

This implies the following observations.

Observation 2.3. *Each pair of non-adjacent vertices of an underlying push clique must have at least two common neighbors.*

Observation 2.4. *An underlying push clique has diameter at most 2.*

Also if an underlying push clique is not a complete graph, then a pair of non-adjacent vertices in it must have at least two common neighbors by Corollary 2.2.