Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the differential and Roman domination number of a graph with minimum degree two

Sergio Bermudo

Department of Economics, Quantitative Methods and Economic History, Universidad Pablo de Olavide, ES-41013 Seville, Spain

ARTICLE INFO

Article history:
Received 29 August 2016
Received in revised form 18 July 2017
Accepted 5 August 2017
Available online xxxx

Keywords:
Differential
Roman domination number

ABSTRACT

Let $G = (V, E)$ be a graph of order n and let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set D. The differential of a vertex set D is defined as $\partial(D) = |B(D)| - |D|$ and the maximum value of $\partial(D)$ for any subset D of V is the differential of G, denoted by $\partial(G)$. A Roman dominating function of G is a function $f : V \to \{0, 1, 2\}$ such that every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is the Roman domination number of G, written $\gamma_R(G)$. Bermudo, et al. proved that these two parameters are complementary with respect to the order n of the graph, that is, $\partial(G) + \gamma_R(G) = n$. In this work we prove that, for any graph G with order $n \geq 9$ and minimum degree two, $\partial(G) \geq \frac{3\gamma(G)}{4}$, consequently, $\gamma_R(G) \leq n - \frac{3\gamma(G)}{4}$, where $\gamma(G)$ is the domination number of G. We also prove that for any graph with order $n \geq 15$, minimum degree two and without any induced tailed 5-cycle graph of seven vertices or tailed 5-cycle graph of seven vertices together with a particular edge, it is satisfied $\partial(G) \geq \frac{5n}{17}$, consequently, $\gamma_R(G) \leq \frac{12n}{17}$.

1. Introduction

Let $G = (V, E)$ be a graph of order n, for every set $D \subseteq V$, let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set D, and let $C(D) = V \setminus (D \cup B(D))$. The *differential of* D is defined as $\partial(D) = |B(D)| - |D|$, and the *differential of a graph* G, written $\partial(G)$, is equal to $\max\{\partial(D)$ such that $D \subseteq V\}$. The study of the mathematical properties of the differential in graphs, together with a variety of other kinds of differentials of a set, started in [14]. In particular, several bounds on the differential of a graph were given. This parameter has also been studied in [1–5,7,17], and the differential in the Cartesian product of graphs has been studied in [18]. The differential of a set D was also considered in [11], where it was denoted by $\eta(D)$.

The Roman domination number is a variant of the domination number suggested by I. Stewart [19], motivated by a problem from military history. A *Roman dominating function* of a graph $G = (V, E)$ is a (total) function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The *weight* of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. This parameter (as well as related ones) has been studied by many authors. We only refer to the papers [8–10,12,13] and the literature quoted therein.

These two parameters were independently studied until 2014, when Bermudo et al. [6] proved that the sum $\gamma_R(G) + \partial(G)$ equals the order of G, so any lower (upper) bound for one will produce an upper (lower) bound for the other. Recall that a set $D \subseteq V$ is a *dominating set* if $B(D) = V \setminus D$. The size of the smallest dominating set is called the *domination number* of

E-mail address: sbernav@upo.es.

http://dx.doi.org/10.1016/j.dam.2017.08.005
0166-218X/

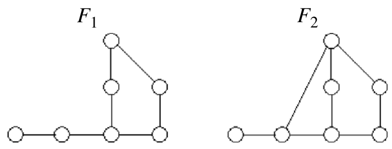

Fig. 1. Graphs F_1 and F_2.

the graph G, written as $\gamma(G)$. Favaron et al. [10] proved that $\gamma_R(G) \leq n - \frac{\gamma(G)}{2}$ for any graph G of order $n \geq 3$, where $\gamma(G)$ is the domination number of G, and Chambers et al. [8] proved that $\gamma_R(G) \leq \frac{8n}{11}$ for any graph G with minimum degree two and order $n \geq 9$. In this work we prove that for any graph G with minimum degree two and order $n \geq 9$, $\partial(G) \geq \frac{3\gamma(G)}{4}$; consequently, $\gamma_R(G) \leq n - \frac{3\gamma(G)}{4}$. We also prove that $\partial(G) \geq \frac{5n}{17}$ for any graph with order $n \geq 15$, minimum degree two, and which does not contain any induced subgraph isomorphic to F_1 or F_2 (see Fig. 1); consequently, $\gamma_R(G) \leq \frac{12n}{17}$. This inequality is true, in particular, when the graph does not contain any induced cycle of five vertices.

We begin by stating some notation and terminology. Let $G = (V, E)$ be a simple graph of *order* $n = |V|$ and *size* $m = |E|$. We denote two adjacent vertices u and v by $u \sim v$ or $\{u, v\} \in E$. For a vertex $u \in V$ we denote $N(v) = \{u \in V : u \sim v\}$ and $N[v] = N(v) \cup \{v\}$, and given a set $S \subseteq V$ we denote $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$. The *degree* of a vertex $v \in V$ will be denoted by $\delta(v) = |N(v)|$. We denote by δ and Δ the *minimum* and *maximum degree* of the graph, respectively. Any graph (V', E') with $V' \subseteq V$ and $E' \subseteq E$ is a *subgraph* of G. A subgraph (V', E') of G is an *induced subgraph* by V', written as $G[V']$, if $E' = \{\{u, v\} \in E : u, v \in V'\}$.

Notice that for every graph G with connected components $G_1, \ldots, G_k$, $\partial(G) = \partial(G_1) + \cdots\cdot + \partial(G_k)$. Therefore, we will only consider connected graphs.

2. Preliminary results

Recall that a graph consisting of one central vertex c and d neighbors that in turn have no further neighbors other than c is also known as a *star* (or *d -star*) $S_d = K_{1,d}$, whose center is c. If $X = (V, E)$ is a d-star with center c, then $V \setminus \{c\}$ will be also called *ray vertices* and the edges will be termed *rays*, in order to stay with the picture. We also denote a d-star X by $X = \{c; v_1, \ldots, v_d\}$ to indicate that c is its center and $v_1, \ldots, v_d$ are its ray vertices. We will call a d-star *big* if $d \geq 2$.

Given a graph $G = (V, E)$, a *big-star packing* is given by a vertex-disjoint collection $\mathcal{S} = \{X_i : 1 \leq i \leq k\}$ of (not necessarily induced) big stars X_i, i.e., the graph induced by $V(X_i)$, written $G[V(X_i)]$ for short, contains a d-star with $d = |V(X_i)| - 1 \geq 2$. If $\mathcal{S}$ is a big-star packing of G, we denote this property by $\mathcal{S} \in SP(G)$. We will write $\mathcal{S}(D)$ when we want to specify that D is the set of vertices which are star centers of $\mathcal{S}$. The set $\mathcal{S}_2(D)$ collects all 2-stars from $\mathcal{S}(D)$, and $\mathcal{S}_{\geq t}(D)$ collects all d-stars from $\mathcal{S}(D)$ such that $d \geq t$. For $S \in \mathcal{S}(D)$, let $C(S, D)$ collect all vertices from $C(D)$ that are neighbors of vertices from S. For a collection $\mathcal{S} \subseteq \mathcal{S}(D)$ of stars, let $C(\mathcal{S}, D) = \bigcup_{S\in\mathcal{S}} C(S, D)$. Finally, we denote $C_2(D) = C(D) \setminus C\left(\mathcal{S}_{\geq 3}(D), D\right)$.

For every $\mathcal{S} \in SP(G)$ we write $\partial(\mathcal{S}) = \sum_{S\in\mathcal{S}}(|S| - 2)$ and call this the *differential* of the big-star packing $\mathcal{S}$. In [3] it was proved that $\partial(G) = \max\{\partial(\mathcal{S}) : \mathcal{S} \in SP(G)\}$. We call a star packing $\mathcal{S} \in SP(G)$ a *differential (star) packing* if it assumes the differential of the graph, i.e., if $\partial(\mathcal{S}) = \partial(G)$. Let $\partial SP(G)$ collect all differential packings of G. We know that given a graph, the set D giving the biggest differential is not, in general, unique. But, even if we fix the set D such that $\partial(D) = \partial(G)$, we can find different big-star packings $\mathcal{S}(D)$ and $\mathcal{S}'(D)$ such that $\partial(\mathcal{S}(D)) = \partial(\mathcal{S}'(D)) = \partial(G)$. Since we will follow the techniques used in [3–5], we will enumerate here some conditions to choose the set D and the big-star packing $\mathcal{S}(D)$ which we will work with. We will also summarize all the results and statements contained in [3] which we will need in our proofs.

We consider the following sets

$$\mathcal{D}_1(G) = \{D \subseteq V : \text{ there exists } \mathcal{S}(D) \in \partial SP(G)\},$$

and

$$\mathcal{D}_2(G) = \{D \in \mathcal{D}_1(G) : \forall D' \in \mathcal{D}_1(G)\ (|D'| \leq |D|)\}.$$

Throughout this paper, if $D \in \mathcal{D}_1(G)$, $\mathcal{S}(D)$ will denote a big-star packing in $\partial SP(G)$. Most of the following results were proved in [3], but since we have changed some notations we will include the proofs of those which have been modified.

Lemma 2.1 ([3]). *If $D \in \mathcal{D}_1(G)$, then the induced graph $G[C(D)]$ decomposes into K_1- and K_2-components.*

Lemma 2.2 ([3]). *If $D \in \mathcal{D}_2(G)$, then any vertex x in $B(D)$ has at most one neighbor in $C(D)$.*

Lemma 2.3 ([3]). *If $D \in \mathcal{D}_2(G)$ and $S \in \mathcal{S}(D)$ with $|V(S)| \geq 4$, then no $x \in V(S) \setminus D$ is neighbor of a K_2-component in $G[C(D)]$.*

Please cite this article in press as: S. Bermudo, On the differential and Roman domination number of a graph with minimum degree two, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.08.005.