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On (1, 2)-step competition graphs of bipartite tournaments

Jihoon Choi, Soogang Eoh, Suh-Ryung Kim *, Sojung Lee

Department of Mathematics Education, Seoul National University, Seoul 151-742, Republic of Korea

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ABSTRACT

In this paper, we study (1, 2)-step competition graphs of bipartite tournaments. A bipartite tournament is an orientation of a complete bipartite graph. We show that the (1, 2)-step competition graph of a bipartite tournament has at most one non-trivial component or consists of exactly two complete components of size at least three and, especially in the former, the diameter of the nontrivial component is at most three if it exists. Based on this result, we show that, among the connected non-complete graphs which are triangle-free or the cycles of which are edge-disjoint, $K_{1,4}$ is the only graph that can be represented as the (1, 2)-step competition graph of a bipartite tournament. We also completely characterize the complete graphs and the disjoint unions of complete graphs which can be represented as the (1, 2)-step competition graph of a bipartite tournament. Finally we present the maximum number of edges and the minimum number of edges which the (1, 2)-step competition graph of a bipartite tournament might have.

1. Introduction

In this paper, all the graphs and digraphs are assumed to be finite and simple. Given a digraph D and a vertex v of D, we define $N_D^+(v) = \{u \in V(D) \mid (v, u) \in A(D)\}$, $N_D^-(v) = \{u \in V(D) \mid (u, v) \in A(D)\}$, $d_D^+(v) = |N_D^+(v)|$, and $d_D^-(v) = |N_D^-(v)|$. For vertices x and y in a digraph D, $d_D(x, y)$ denotes the number of arcs in a shortest directed path from x to y in D if it exists. For positive integers i and j, the (i, j)-*step competition graph* of a digraph D, denoted by $C_{i,j}(D)$, is a graph on $V(D)$ where $uv \in E(C_{i,j}(D))$ if and only if there exists a vertex w distinct from u and v such that either $d_{D-v}(u, w) \leq i$ and $d_{D-u}(v, w) \leq j$ or $d_{D-u}(v, w) \leq i$ and $d_{D-v}(u, w) \leq j$. The (1, 1)-step competition graph of a digraph D is the competition graph of D. Given a digraph D, the *competition graph* of D, denoted by $C(D)$, is the graph having vertex set $V(D)$ and edge set $\{uv \mid (u, w) \in A(D), (v, w) \in A(D) \text{ for some } w \in V(D)\}$. Cohen [4] introduced the notion of competition graph while studying predator–prey concepts in ecological food webs. Cohen's empirical observation that real-world competition graphs are usually interval graphs had led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs. In the same vein, various variants of competition graph have been introduced and studied, one of which is the notion of (i, j)-step competition introduced by Factor and Merz [5] (see [1,2,9,10,14,15] for other variants of competition graph). For recent work on this topic, see [3,6,7,11–13].

Factor and Merz [5] studied the (1, 2)-step competition graphs of tournaments. Zhang and Li [16] and Zhang et al. [17] studied the (1, 2)-step competition graphs of round digraphs. On the other hand, Kim et al. [8] studied the competition graphs of orientations of complete bipartite graphs. In this paper, we study the (1, 2)-step competition graphs of orientations of complete bipartite graphs, which is a natural extension of their results.

An orientation of a complete bipartite graph is sometimes called a *bipartite tournament* and we use whichever of the two terms is more suitable for a given situation throughout this paper.

* Corresponding author.
E-mail addresses: gaouls@snu.ac.kr (J. Choi), mathfish@snu.ac.kr (S. Eoh), srkim@snu.ac.kr (S.-R. Kim), sj023@snu.ac.kr (S. Lee).

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In Section 2, we derive useful properties of $(1, 2)$-step competition graphs of bipartite tournaments. In Section 3, based on the results obtained in Section 2, we show that, among the connected non-complete graphs which are triangle-free or the cycles of which are edge-disjoint, $K_{1,4}$ is the only graph that can be represented as the $(1, 2)$-step competition graph of a bipartite tournament. We also completely characterize a complete graph and the disjoint union of complete graphs, respectively, which can be represented as the $(1, 2)$-step competition graph of a bipartite tournament. In Section 4, we present the maximum number of edges and the minimum number of edges which the $(1, 2)$-step competition graph of an orientation of $K_{m,n}$ might have.

2. Properties of (1, 2)-step competition graphs of bipartite tournaments

For a digraph D, we say that vertices u and v in D $(1, 2)$-*compete* provided there exists a vertex w distinct from u, v and satisfying one of the following:

- there exist an arc (u, w) and a directed (v, w)-walk of length 2 not traversing u;
- there exist a directed (u, w)-walk of length 2 not traversing v and an arc (v, w).

We call w in the above definition a $(1, 2)$-*step common out-neighbor* of u and v. It is said that two vertices *compete* if they have a common out-neighbor. Thus, $uv \in E(C_{1,2}(D))$ provided u and v compete or $(1, 2)$-compete.

An edge in the $(1, 2)$-step competition graph of a digraph D is said to *be induced by competition* (resp. $(1, 2)$-*step competition*) if there exists a common out-neighbor (resp. $(1, 2)$-step common out-neighbor) of the ends of the edge in D.

From the definition of the $(1, 2)$-step competition graph of a digraph, we make the following simple but useful observation.

Proposition 1. *In the $(1, 2)$-step competition graph of a digraph D, a non-isolated vertex has an out-neighbor in D.*

We present results which will play a key role throughout this paper.

Proposition 2. *Let u and v be adjacent vertices in the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$. Then u and v belong to the same partite set if and only if they compete in D.*

Proof. Since u and v are adjacent in the $(1, 2)$-step competition graph of D, u and v compete or $(1, 2)$-compete in D. If u and v belong to the same partite set, then u and v cannot $(1, 2)$-compete in D as they cannot be connected by a walk of length 3 in $K_{m,n}$. Thus, if u and v belong to the same partite set, then they compete in D. Conversely, if u and v belong to different partite sets, then they cannot compete in D since no vertex of D can be a common out-neighbor of u and v in D. □

The following corollary is equivalent to Proposition 2.

Corollary 3. *Let u and v be adjacent vertices in the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$. Then u and v belong to distinct partite sets if and only if they $(1, 2)$-compete in D.*

By Corollary 3, two vertices belonging to distinct partite sets of $K_{m,n}$ can only $(1, 2)$-compete in any of its orientations. The following theorem characterizes a pair of vertices which belong to distinct partite sets of $K_{m,n}$ and $(1, 2)$-compete.

Theorem 4. *Let u and v be vertices belonging to distinct partite sets of a bipartite tournament D. Then u and v $(1, 2)$-compete in D if and only if u (resp. v) has an out-neighbor different from v (resp. u).*

Proof. Let (V_1, V_2) be a bipartition of D. Without loss of generality, we may assume that $u \in V_1$ and $v \in V_2$. The 'only if' part is obviously true. To show the 'if' part, suppose that x is an out-neighbor of u different from v and y is an out-neighbor of v different from u. Since D is a bipartite tournament, either (x, y) or (y, x) is an arc in D. If (x, y) (resp. (y, x)) is an arc, then there exists a directed walk $u \to x \to y$ (resp. $v \to y \to x$) and therefore u and v $(1, 2)$-compete. □

The following corollary is to be quoted frequently in the rest of this paper.

Corollary 5. *Let D be a bipartite tournament. Suppose that $(u, v) \in A(D)$ and the outdegree of v is at least one. Then u and v are adjacent in the $(1, 2)$-step competition graph of D if and only if u has at least two out-neighbors in D.*

Proof. Since (u, v) is an arc in D, u and v belong to distinct partite sets of D. For the same reason, u has an out-neighbor different from v if and only if u has at least two out-neighbors in D. Moreover, by the hypothesis, v has an out-neighbor. Since (u, v) is an arc in D, it is different from u. Therefore u (resp. v) has an out-neighbor different from v (resp. u) if and only if u has at least two out-neighbors in D. Thus the corollary follows from Theorem 4. □

Corollary 6. *Let G be the $(1, 2)$-step competition graph of an orientation D of $K_{m,n}$ with a bipartition (V_1, V_2). Then each vertex has outdegree at least two in D if and only if the edges of G not belonging to $G[V_1] \cup G[V_2]$ induce $K_{m,n}$.*