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Tractability, hardness, and kernelization lower bound for and/or graph solution

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ABSTRACT

And/or graphs are well-known data structures with several applications in many fields of computer science, such as Artificial Intelligence, Distributed Systems, Software Engineering, and Operations Research. An and/or graph is an acyclic digraph G containing a single source vertex s, where every vertex $v \in V(G)$ has a label $f(v) \in \{\texttt{and}, \texttt{or}\}$. In an and/or graph, (weighted) edges represent dependency relations between vertices: a vertex labeled and depends on all of its out-neighbors, while a vertex labeled or depends on only one of its out-neighbors. A solution subgraph H of an and/or graph G is a subdigraph of G containing its source vertex and such that if an and-vertex (resp. or-vertex) is included in H then all (resp. one) of its out-edges must also be included in H. In general, solution subgraphs represent feasible solutions of problems modeled by and/or graphs. The optimization problem associated with an and/or graph G consists of finding a *minimum weight solution subgraph* H of G, where the weight of a solution subgraph is the sum of the weights of its edges. Because of its wide applicability, in this work we develop a multivariate investigation of this optimization problem. In a previous paper (Souza et al., 2013) we have analyzed the complexity of such a problem under various aspects, including parameterized versions of it. However, the main open question has remained open: Is the problem of finding a solution subgraph of weight at most k (where k is the parameter) in FPT? In this paper we answer negatively to this question, proving the W[1]-hardness of the problem, and its W[P]-completeness when zero-weight edges are allowed. We also show that the problem is fixed-parameter tractable when parameterized by the tree-width, but it is W[SAT]-hard with respect to the clique-width and k as aggregated parameters. In addition, we show that when the out-edges of each or-vertex have all the same weight (a condition very common in practice), the problem becomes fixed-parameter tractable by the clique-width. Finally, using a framework developed by Bodlaender *et al.* (2009) and Fortnow and Santhanam (2011), based upon the notion of compositionality, we show that the tractable cases do not admit a polynomial kernel unless $NP \subseteq coNP/poly$, even restricted to instances without or-vertices with out-degree greater than two.

1. Introduction

In this paper, we consider the complexity of a problem associated with relevant data structures, the so-called *and/or graphs*. An and/or graph is an acyclic digraph containing a single source vertex (that reaches all other vertices by directed paths), such that every vertex $v \in V(G)$ has a label $f(v) \in \{\texttt{and}, \texttt{or}\}$. In such digraphs, edges represent dependency

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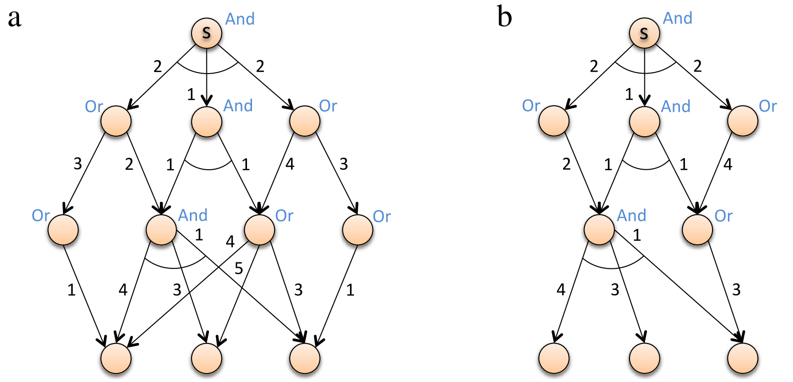

Fig. 1. (a) An and/or graph; (b) An optimal solution subgraph of the graph in (a).

relations between vertices: a vertex labeled and depends on all of its out-neighbors (conjunctive dependency), while a vertex labeled or depends on only one of its out-neighbors (disjunctive dependency). In this paper, we denote by O_{v_i} the set of out-neighbors of vertex v_i. Fig. 1 shows examples of and/or graphs, where and-vertices have an arc around its out-edges.

And/or graphs arise in several fields and applications, such as Artificial Intelligence [4,28,30], Automation and Robotics [10,15,21], Distributed Systems [3], Game Theory [22], Hypergraph Applications [20], Computational Logics [23], Operations Research [27,32], Scheduling [2], and Software Engineering [11,25], to name a few.

The main optimization problem associated with and/or graphs is formally defined below.

MIN-AND/OR
Instance: An and/or graph $G = (V, E)$ containing a single source vertex s, and such that each edge e of G has an integer weight $\tau(e) > 0$.
Goal: Determine the minimum weight of a subdigraph $H = (V', E')$ of G (*solution subgraph*) satisfying the following properties:

- $s \in V'$;
- if a vertex v is in V' and $f(v)$=and then every out-edge of v belongs to E';
- if a non-sink vertex v is in V' and $f(v)$=or then exactly one out-edge of v belongs to E'.

MIN-AND/OR^0 is a generalization of MIN-AND/OR where zero-weight edges are allowed.

In 1974, Sahni [29] showed that MIN-AND/OR is NP-hard; however, there are some cases for which it can be solved in polynomial time (for details, see [31]).

We denote by MIN-AND/OR(k) the parameterized version of MIN-AND/OR which asks whether there is a solution subgraph of weight at most k. This approach is justified by the fact that many applications are concerned with satisfying a low cost limit. In a previous paper [31], we investigate some variants of MIN-AND/OR(k): (a) MIN-AND/OR(k, r), the parameterized version where every or-vertex of the input graph has at most r out-edges with the same weight; (b) MIN-AND/OR^0 (k), the parameterized version applied to and/or graphs where zero-weight edges are allowed. In [31] we prove that MIN-AND/OR(k, r) is in FPT, whereas MIN-AND/OR^0 (k) is W[2]-hard. However, the main question of classifying MIN-AND/OR(k) has remained open up to now. In this paper we close this question by proving that MIN-AND/OR(k) is W[1]-hard and in W[P]; we also show that MIN-AND/OR^0 (k) is in fact W[P]-complete. In addition, some fixed-parameter tractable cases are presented, and a proof that such tractable cases do not admit a polynomial kernel, unless $NP \subseteq coNP/poly$, is given. The latter condition, if true, would imply an unlikely collapse of the polynomial hierarchy to the third level ($PH = \Sigma_p^3$).

The remainder of this work is organized as follows. In Section 2 we show the W[P]-completeness of MIN-AND/OR^0 (k). In Section 3 we prove that MIN-AND/OR(k) is W[1]-hard. In Section 4, we consider the tree-width and the clique-width as parameters. Finally, in Section 5, we present kernelization lower bounds to some cases.

2. W[P]-Completeness

The class W[P] is defined as the class of parameterized problems fpt-reducible to WEIGHTED CIRCUIT SATISFIABILITY [17].

WEIGHTED CIRCUIT SATISFIABILITY
Instance: A decision circuit C.
Parameter: A positive integer k.
Question: Does C have a satisfying assignment of weight k?

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