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Inapproximability results and bounds for the Helly and Radon numbers of a graph

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ABSTRACT

Let $\mathcal{C}$ be a convexity on a set X and denote the convex hull of $S \subseteq X$ in $\mathcal{C}$ by $H(S)$. The Helly number (Radon number) of $\mathcal{C}$ is the minimum integer k such that, for every $S \subseteq X$ with at least $k+1$ elements, it holds $\bigcap_{v \in S} H(S \setminus \{v\}) \neq \varnothing$ (there is a bipartition of S into sets S_1 and S_2 with $H(S_1) \cap H(S_2) \neq \varnothing$). In this work, we show that there is no approximation algorithm for the Helly or the Radon number of a graph G of order n in the geodetic convexity to within a factor $n^{1-\varepsilon}$ for any $\varepsilon > 0$, unless P $=$ NP, even if G is bipartite. Furthermore, we present upper bounds for both parameters in the geodetic convexity of bipartite graphs and characterize the families of graphs achieving the bound for the Helly number.

1. Introduction

Given a finite set X, a family $\mathcal{C}$ of subsets of X is a *convexity on* X if $\varnothing, X \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections [15]. Every member of $\mathcal{C}$ is said to be a *convex set of* $\mathcal{C}$. The *convex hull of* $S \subseteq X$ *in* $\mathcal{C}$, $H(S)$, is the minimum convex set of $\mathcal{C}$ containing S. The *Helly number of* $\mathcal{C}$ is the minimum integer h such that, for every set $S \subseteq X$ with at least $h+1$ elements, it holds $\bigcap_{v \in S} H(S \setminus \{v\}) \neq \varnothing$. The *Radon number of* $\mathcal{C}$ is the minimum integer r such that, for every set $S \subseteq X$ of size at least $r+1$, there is a bipartition of S into sets S_1 and S_2 satisfying $H(S_1) \cap H(S_2) \neq \varnothing$.

The classical results that originated the Helly and Radon numbers were published in the beginning of the last century [8,13] and described properties for convex sets in the Euclidian space which are similar to the concepts that they gave rise. After that, the theory of abstract convexities was widespread [5–7,14] and one of its fundamental results says that, for any abstract convexity, the Helly number h is upper bounded by the Radon number r, more specifically, $h \leq r$ [11].

On graphs, the most studied convexities have their convex sets defined by means of a set of paths. In the geodetic convexity [12], the set of "shortest paths" is considered, i.e., a set $S \subseteq V(G)$ is convex if the vertices of every shortest path of G joining two vertices of S are contained in S. Other widely studied graph convexities are the monophonic convexity [1,4] and the P_3 convexity [9], where the set of "induced paths" and "paths of order three" are considered in the definition of convex sets, respectively. It is also known that on the monophonic convexity the Helly number of any graph equals the size of its maximum clique [10].

Recently, it was proved in [3] that the problem of deciding if a graph has a Radon number at most k in the P_3 convexity is NP-hard even for split graphs. Furthermore, an upper bound for the Radon number on the P_3 convexity for general graphs was given in [2].

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In this work, we show that there is no approximation algorithm for computing the Helly or the Radon numbers in the geodetic convexity of a graph G of order n to within a factor $n^{1-\epsilon}$ for any $\varepsilon > 0$, even if G is bipartite, unless P = NP. Since no polynomial time algorithms are probable for these problems, turns out interesting to look for bounds. It is easy to see that the clique number of a graph G is a lower bound to the Helly and the Radon numbers of G. This implies that the Helly and the Radon numbers of a complete graph with n vertices is n. Therefore, n is a tight upper bound for the Helly and Radon numbers of general graphs. It becomes then interesting to look for upper bounds on graphs with small clique number, which is the case of bipartite graphs, where the problems of determining these parameters do not seem to be easier than the general case. In this sense, we present upper bounds for both parameters in the geodetic convexity of bipartite graphs and characterize the family of graphs achieving the bound for the Helly number.

The text is organized as follows. In the rest of this section we give useful definitions and properties. Next, in Section 2, we present the inapproximability results and Section 3 is devoted to the bounds related to the Helly and Radon numbers of a bipartite graphs.

1.1. Definitions and properties

Equivalent definitions to the ones given previously for the Helly and Radon numbers can be obtained in the following way. Consider a convexity $\mathcal{C}$ on a set X and $S = \{v_1, \ldots, v_k\} \subseteq X$. We say that S is *Helly independent* if $\bigcap_{i=1}^{k} H(S \setminus \{v_i\}) = \varnothing$. Then, the Helly number of $\mathcal{C}$ is the cardinality of a maximum Helly independent subset of X. We say that S is *Radon independent* if every partition (S_1, S_2) of S satisfies $H(S_1) \cap H(S_2) = \varnothing$. Hence, the Radon number of $\mathcal{C}$ is the cardinality of a maximum Radon independent subset of X.

Our results concern the geodetic convexity on the vertex set of a finite, simple, and connected graph. Then, from now on, when we mention the Helly (Radon) number of a graph, we will be referring to the Helly (Radon) number of the geodetic convexity of such a graph.

For a graph G, denote its vertex set, edge set, Helly number, and Radon number by $V(G)$, $E(G)$, $h(G)$ and $r(G)$, respectively. The *order of* G is the cardinality of $V(G)$. A shortest path joining two vertices u and v of G is said to be a (u, v)-*geodesic of* G. The number of edges in a (u, v)-geodesic is the *distance*, $d(u, v)$, between u and v. The *(geodetic) interval of a set* $S \subseteq V(G)$, denoted by $I(S)$, is formed by S and every vertex lying in some geodesic between two vertices of S. When S contains only two vertices, say u, v, sometimes we will write $I(u, v)$ instead of $I(S) = I(\{u, v\})$ for reasons of clarity. Observe that the convex hull of S can be obtained by recursively applying the interval function until a convex set be obtained. If $H(S) = V(G)$, then S is said to be a *hull set of* G.

A set of vertices of a graph G is a *clique* (an *independent set*) if every two vertices of it are adjacent (non-adjacent). We say that G is *bipartite* if $V(G)$ can be partitioned into two independent sets; and that G is a *complete graph* if $V(G)$ is a clique. A complete graph of order n can be denoted by K_n. The *neighbourhood of vertex* $v \in V(G)$ is $N(v) = \{u : uv \in E(G)\}$. Finally, given two (disjoint) graphs G_1 and G_2, the *union of* G_1 *and* G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

We conclude this section presenting useful tools.

Lemma 1. *Let G be a graph. If $S \subseteq V(G)$ is a Helly (or a Radon) independent set with at least 3 vertices, then the subgraph of G induced by S is a union of complete graphs. Furthermore, if G is bipartite, then S is an independent set.*

Proof. Let $S \subseteq V(G)$ be any set such that the subgraph of G induced by S is not a union of complete graphs. Then, there exist vertices $u, v, w \in S$ such that $u, v \notin E(G)$ and $w \in N(u) \cap N(v)$. Hence $w \in I(u, v)$. This implies that $(\{w\}, S \setminus \{w\})$ is a Radon partition of S. Then, S is not a Radon independent set.

Now, consider that G is bipartite with bipartition (A, B) and let $S \subseteq V(G)$ containing adjacent vertices a and b. Without loss of generality, we can say that $a \in A$ and there exists $a' \in (A \cap S) \setminus \{a\}$. It is clear that $d(a', b) - 1 \leq d(a', a) \leq d(a', b) + 1$. This implies that $a \in I(a', b)$ or $b \in I(a', a)$. As we have seen above, this means that S is not a Radon independent set.

To complete the proof it suffices to observe that every Helly independent set S is also Radon independent, because $H(S_1) \cap H(S_2) \subseteq \bigcap_{v \in S} H(S \setminus \{v\})$ for any partition (S_1, S_2) of S. □

Lemma 2. *Let u, v, and w be vertices of a bipartite graph G. If $uv \in E(G)$, then either $u \in I(v, w)$ or $v \in I(u, w)$.*

Proof. First, we claim that $d(u, w) \neq d(v, w)$. Thus, suppose for contradiction that $d(u, w) = d(v, w)$. Let P_{uw} and P_{vw} be geodesics beginning at u and v, respectively, and finishing at w. Denote by w' the most distant vertex of w that belongs to both P_{uw} and P_{vw}. Now, let $P_{uw'}$ and $P_{vw'}$ be the subpaths of P_{uw} and P_{vw} beginning at u and v, respectively, and finishing at w'. It is clear that $P_{uw'}$ and $P_{vw'}$ have the same length and, therefore, the concatenation of $P_{uw'}$, $P_{vw'}$, and uv form an odd cycle. This is a contradiction because G is a bipartite graph. Hence, $d(u, w) \neq d(v, w)$ as claimed. Now, it remains to notice that if $d(u, w) < d(v, w)$, then $u \in I(v, w)$ and $v \notin I(u, w)$. □