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# Eccentricity approximating trees[☆]

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ABSTRACT

Using the characteristic property of chordal graphs that they are the intersection graphs of subtrees of a tree, Erich Prisner showed that every chordal graph admits an eccentricity 2-approximating spanning tree. That is, every chordal graph $G$ has a spanning tree $T$ such that $ecc_T(v) - ecc_G(v) \leq 2$ for every vertex $v$, where $ecc_G(v)$ ($ecc_T(v)$) is the eccentricity of a vertex $v$ in $G$ (in $T$, respectively). Using only metric properties of graphs, we extend that result to a much larger family of graphs containing among others chordal graphs, the underlying graphs of 7-systolic complexes and plane triangulations with inner vertices of degree at least 7. Furthermore, based on our approach, we propose two heuristics for constructing eccentricity $k$-approximating trees with small values of $k$ for general unweighted graphs. We validate those heuristics on a set of real-world networks and demonstrate that all those networks have very good eccentricity approximating trees.


## 1. Introduction

All graphs $G = (V, E)$ occurring in this paper are connected, finite, unweighted, undirected, loopless and without multiple edges. The *length* of a path from a vertex $v$ to a vertex $u$ is the number of edges in the path. The *distance* $d_G(u, v)$ between two vertices $u$ and $v$ is the length of a shortest path connecting $u$ and $v$ in $G$. If no confusion arises, we will omit subindex $G$. The interval $I(u, v)$ between $u$ and $v$ consists of all vertices on shortest $(u, v)$-paths, that is, it consists of all vertices (metrically) between $u$ and $v$: $I(u, v) = \{x \in V : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$. The *eccentricity* $ecc_G(v)$ of a vertex $v$ in $G$ is defined by $\max_{u \in V} d_G(u, v)$, i.e., it is the distance to a most distant vertex. The *diameter* of a graph is the maximum over the eccentricities of all vertices: $diam(G) = \max_{u \in V} ecc_G(u) = \max_{u,v \in V} d_G(u, v)$. The *radius* of a graph is the minimum over the eccentricities of all vertices: $rad(G) = \min_{u \in V} ecc_G(u)$. The set of vertices with minimum eccentricity forms the *center* $C(G)$ of a graph $G$, i.e., $C(G) = \{u \in V : ecc_G(u) = rad(G)\}$. Recall that for every graph $G$, $diam(G) \leq 2rad(G)$ holds.

A spanning tree $T$ of a graph $G$ with $d_T(u, v) - d_G(u, v) \leq k$, for all $u, v \in V$, is known as an *additive tree k-spanner* of $G$ [20] and, if it exists for a small integer $k$, then it gives a good approximation of all distances in $G$ by the distances in $T$. Many optimization problems involving distances in graphs are known to be NP-hard in general but have efficient solutions in simpler metric spaces, with well-understood metric structures, including trees. A solution to such an optimization problem obtained for a tree spanner $T$ of $G$ usually serves as a good approximate solution to the problem in $G$.

E. Prisner in [31] introduced the new notion of eccentricity approximating spanning trees. A spanning tree $T$ of a graph $G$ is called an *eccentricity k-approximating spanning tree* if $ecc_T(v) - ecc_G(v) \leq k$ holds for all $v \in V$. Such a tree tries to approximately preserve only distances from each vertex $v$ to its most distant vertices and can tolerate larger increases to nearby vertices. They are important in applications where vertices measure their degree of centrality by means of

☆ Results of this paper were partially presented at the WG'16 conference Dragan et al. (2016) [11].

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their eccentricity and would tolerate a small surplus to the actual eccentricities [31]. Note also that Nandakumar and Parthasarasthy considered in [26] eccentricity-preserving spanning trees (i.e., eccentricity 0-approximating spanning trees) and showed that a graph $G$ has an eccentricity 0-approximating spanning tree if and only if: (a) either $diam(G) = 2rad(G)$ and $|C(G)| = 1$, or $diam(G) = 2rad(G) - 1$, $|C(G)| = 2$, and those two center vertices are adjacent; (b) every vertex $u \in V \setminus C(G)$ has a neighbor $v$ such that $ecc_G(v) < ecc_G(u)$.

Every additive tree $k$-spanner is clearly eccentricity $k$-approximating. Therefore, eccentricity $k$-approximating spanning trees can be found in every interval graph for $k = 2$ [20,22,30], and in every asteroidal-triple-free graph [20], strongly chordal graph [3] and dually chordal graph [3] for $k = 3$. On the other hand, although for every $k$ there is a chordal graph without a tree $k$-spanner [20,30], yet as Prisner demonstrated in [31], every chordal graph has an eccentricity 2-approximating spanning tree, i.e., with the slightly weaker concept of eccentricity-approximation, one can be successful even for chordal graphs.

Unfortunately, the method used by Prisner in [31] heavily relies on a characteristic property of chordal graphs (*chordal graphs are exactly the intersection graphs of subtrees of a tree*) and is hardly extendable to larger families of graphs.

In this paper we present a new proof of the result of [31] using only metric properties of chordal graphs. This allows us to extend the result to a much larger family of graphs which includes not only chordal graphs but also other families of graphs known from the literature.

It is known [5,35] that every chordal graph satisfies the following two metric properties:

$\alpha_1$-metric: if $v \in I(u, w)$ and $w \in I(v, x)$ are adjacent, then $d_G(u, x) \geq d_G(u, v) + d_G(v, x) - 1 = d_G(u, v) + d_G(w, x)$.
triangle condition: for any three vertices $u, v, w$ with $1 = d_G(v, w) < d_G(u, v) = d_G(u, w)$ there exists a common neighbor $x$ of $v$ and $w$ such that $d_G(u, x) = d_G(u, v) - 1$.

A graph $G$ satisfying the $\alpha_1$-metric property is called an $\alpha_1$-*metric graph*.[1] If an $\alpha_1$-metric graph $G$ satisfies also the triangle condition then $G$ is called an $(\alpha_1, \Delta)$-*metric graph*. We prove that every $(\alpha_1, \Delta)$-metric graph $G = (V, E)$ has an eccentricity 2-approximating spanning tree and that such a tree can be constructed in $\mathcal{O}(|V||E|)$ total time. As a consequence, we get that the underlying graph of every 7-systolic complex (and, hence, every plane triangulation with inner vertices of degree at least 7 and every chordal graph) has an eccentricity 2-approximating spanning tree.

The paper is organized as follows. In Section 2, we present additional notions and notations and some auxiliary results. In Section 3, some useful properties of the eccentricity function on $(\alpha_1, \Delta)$-metric graphs are described. Our eccentricity approximating spanning tree is constructed and analyzed in Section 4. In Section 5, the algorithm for the construction of an eccentricity approximating spanning tree developed in Section 4 for $(\alpha_1, \Delta)$-metric graphs is generalized and validated on some real-world networks. Our experiments show that all those real-world networks have very good eccentricity approximating trees. Section 6 concludes the paper with a few open questions.

## 2. Preliminaries

For a graph $G = (V, E)$, we use $n = |V|$ and $m = |E|$ to denote the cardinality of the vertex set and the edge set of $G$. We denote an *induced cycle* of length $k$ by $C_k$ (i.e., it has $k$ vertices) and by $W_k$ an *induced wheel* of size $k$ which is a $C_k$ with one extra vertex universal to $C_k$. For a vertex $v$ of $G$, $N_G(v) = \{u \in V : uv \in E\}$ is called the *open neighborhood*, and $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of $v$. The distance between a vertex $v$ and a set $S \subseteq V$ is defined as $d_G(v, S) = \min_{u \in S} d_G(u, v)$ and the set of furthest (most distant) vertices from $v$ is denoted by $F(v) = \{u \in V : d_G(u, v) = ecc_G(v)\}$.

An induced subgraph of $G$ (or the corresponding vertex set $A$) is called *convex* if for each pair of vertices $u, v \in A$ it includes the interval $I(v, u)$ of $G$ between $u, v$. An induced subgraph $H$ of $G$ is called *isometric* if the distance between any pair of vertices in $H$ is the same as their distance in $G$. In particular, convex subgraphs are isometric. The disk $D(x, r)$ with center $x$ and radius $r \geq 0$ consists of all vertices of $G$ at distance at most $r$ from $x$. In particular, the unit disk $D(x, 1) = N[x]$ comprises $x$ and the neighborhood $N(x)$. For an edge $e = xy$ of a graph $G$, let $D(e, r) := D(x, r) \cup D(y, r)$.

By the definition of $\alpha_1$-metric graphs clearly, such a graph cannot contain any isometric cycles of length $k > 5$ and any induced cycle of length 4. The following results characterize $\alpha_1$-metric graphs and the class of chordal graphs within the class of $\alpha_1$-metric graphs. Recall that a graph is *chordal* if all its induced cycles are of length 3.

**Theorem 1** (*[35]*). *$G$ is a chordal graph if and only if it is an $\alpha_1$-metric graph not containing any induced subgraphs isomorphic to cycle $C_5$ and wheel $W_k$, $k \geq 5$.*

**Theorem 2** (*[35]*). *$G$ is an $\alpha_1$-metric graph if and only if all disks $D(v, k)$ ($v \in V$, $k \geq 1$) of $G$ are convex and $G$ does not contain the graph $W_6^{++}$ (see Fig. 1) as an isometric subgraph.*

**Theorem 3** (*[13,32]*). *All disks $D(v, k)$ ($v \in V$, $k \geq 1$) of a graph $G$ are convex if and only if $G$ does not contain isometric cycles of length $k > 5$, and for any two vertices $x, y$ the neighbors of $x$ in the interval $I(x, y)$ are pairwise adjacent.*

[1] A more general concept of $\alpha_i$-metric was introduced in [35], however, in this paper, we are interested only in the case when $i = 1$.