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Note

Counterexamples to the conjecture on orientations of graphs with minimum Wiener index☆

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ABSTRACT

In Knor et al. (2016) authors conjectured that for a graph G, $W_{\min}(G)$ is achieved for a $\chi(G)$-coloring-induced orientation. They also proved the conjecture holds for bipartite graphs, complete graphs, prisms and Petersen graph.

However, we find that there exists a graph G such that any minimum Wiener index orientation of G is not $\chi(G)$-coloring-induced. This means the conjecture does not hold in general. Furthermore, we explore some graphs which do not satisfy the conjecture.

1. Introduction

A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges. For any two vertices $u, v \in V(G)$, the distance between them, denoted by $d_G(u, v)$, is the length of a shortest path connecting them in G.

A digraph D is given by a set of vertices $V(D)$ and a set of ordered pairs of vertices $A(D)$ called directed edges or arcs.

Let G be a graph. The Wiener index of G is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v),$$

where the sum is taken over all unordered pairs of vertices of G. It was introduced by H. Wiener in 1947 to predict the boiling point of paraffin. It has been considered much later by graph theorists and it was studied under other names. Many papers also deal with the average distance, defined as $\mu(G) = W(G)/\binom{n}{2}$.

The Wiener index of a graph is considered as one of the most popular molecular descriptors. Some recent works on mathematical aspects of Wiener index refer to survey papers [2,5,8] and references therein.

The first results on the Wiener index of a digraph are due to F. Harary whose investigation was motivated by certain sociometric problems. The Wiener index of a digraph was considered indirectly also through the study of the average distance [1,3].

Recently M. Knor, R. Škrekovski and A. Tepeh [7] extended the concept of Wiener index to digraphs which are not-necessarily strongly connected, following the convention that the distance from u to v is zero if there is no directed path from vertex u to vertex v. This extension could be applicable in the topics of directed large networks.

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Let D be a digraph. We use $d_D(u, v)$ to denote the distance from u to v in D, which is the length of a shortest directed path from u to v. The Wiener index of D is defined as

$$W(D) = \sum_{(u,v)\in V(D)\times V(D)} d_D(u, v),$$

where the sum is taken over all ordered pairs of vertices of D. Note that $d_D(u, v) = 0$ if there is no directed path from u to v and the distances $d_D(u, v)$ and $d_D(v, u)$ may be different.

In [4,6], M. Knor et al. characterized tournaments with the maximal and the second maximal Wiener indices, and obtained that the orientation of a given Theta-graph which yields the maximum Wiener index is not necessarily strongly connected. They also generalized the Wiener theorem, as well as a relation between the Wiener index and betweenness centrality to directed graphs and formulated some conjectures about orientations of graphs which achieve the extremal values of Wiener index [7].

Let G be a graph. An orientation of G is an assignment of a direction to each edge, turning the initial graph into a digraph. We use the symbol $\mathbb{D}(G)$ to specify all orientations of G and the symbol $W_{\min}(G)$ to specify the minimum Wiener index of $\mathbb{D}(G)$. If an orientation of G achieves the minimum Wiener index $W_{\min}(G)$, we call this orientation a minimum Wiener index orientation of G.

Let D be a digraph. If uv is an arc in D, we say that u dominates v. The vertices which dominate a vertex v are its in-neighbors, those which are dominated by the vertex are its out-neighbors. These sets are denoted by $N_D^-(v)$ and $N_D^+(v)$, respectively. The in-degree $d_D^-(v)$ of a vertex v is the number of its in-neighbors, $d_D^-(v) = \left|N_D^-(v)\right|$. Similarly, we define the term out-degree, denoted by $d_D^+(v)$, as $\left|N_D^+(v)\right|$. A vertex of in-degree (out-degree) zero is called a source (sink). The degree of v is $d_D(v) = d_D^+(v) + d_D^-(v)$.

For above notations, we sometimes omit the index D when no confusion is likely.

A k-vertex-coloring of a graph G, or simply a k-coloring, is an assignment of k colors to its vertices. A k-coloring of G is proper (or G is k-colorable) if no two adjacent vertices are assigned the same color. The minimum k for which G is k-colorable is called the chromatic number of G, and is denoted by $\chi(G)$.

An orientation of a graph G is called k-coloring-induced, if it is obtained from a proper k-coloring of G such that each edge is oriented from the end-vertex with the bigger color to the end-vertex with the smaller color.

Based on the concept of k-coloring-induced orientation, recent paper [7] gave the following χ-coloring-induced conjecture.

Conjecture 1.1 ([7]). *$W_{\min}(G)$ is achieved for a $\chi(G)$-coloring-induced orientation.*

Knor et al. studied orientations of some graphs and showed that Conjecture 1.1 holds for bipartite graphs, complete graphs and prisms. By computer it was tested also for the Petersen graph. They thought if Conjecture 1.1 is not true in general, it may be satisfied at least for 3-colorable graphs.

However, we find that there are some counterexamples to Conjecture 1.1 even for 3-colorable graphs. In Section 2, we introduce some useful concepts and a fundamental principle. Then in Section 3 we give a counterexample to Conjecture 1.1. In Section 4, we make further efforts to explore some graphs which do not satisfy Conjecture 1.1.

2. Preliminaries

We will introduce two new concepts which are very useful in this work.

2.1. *I-vertex*

In a digraph D, a vertex that is neither a source nor a sink is called I-vertex (internal vertex). If a vertex is not an I-vertex, then we say it is a non-I-vertex. Obviously, if u is an I-vertex, then $d_D^+(u) \times d_D^-(v) \geqslant 1$. Any orientation of an odd cycle has at least one I-vertex. This vertex is a central vertex of a path of length 2 in the orientation. Later we will show that the concept of I-vertex plays an essential role in orientations that achieved the minimum Wiener index.

2.2. *Wiener increment*

Let D be a digraph. For $u \in V(D)$, the Wiener index of u is defined as $w(u) = \sum_{v\in V(D)} d_D(u, v)$. Then the Wiener index of D satisfies $W(D) = \sum_{u\in V(D)} w(u)$.

For $u \in V(D)$, the Wiener increment of u is defined as $\Delta w(u) = w(u) - d_D^+(u)$. It is clear that $\Delta w(u) = \sum_{v\in V(D), d_D(u,v)>1} d_D(u, v)$.