ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Hamiltonian cycles in linear-convex supergrid graphs☆

Ruo-Wei Hung

Department of Computer Science and Information Engineering, Chaoyang University of Technology, Wufeng, Taichung 41349, Taiwan

ARTICLE INFO

Article history:
Received 16 July 2015
Received in revised form 30 January 2016
Accepted 18 April 2016
Available online xxxx

Keywords:
Hamiltonian cycle
Locally connected
Linear-convex supergrid graph
Supergrid graph
Grid graph
Triangular grid graph
Computer sewing machine

ABSTRACT

A supergrid graph is a finite induced subgraph of the infinite graph associated with the two-dimensional supergrid. The supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. The Hamiltonian cycle problem for grid and triangular grid graphs was known to be NP-complete. Recently, we have proved the Hamiltonian cycle problem on supergrid graphs to be NP-complete. The Hamiltonian cycle problem on supergrid graphs can be applied to control the stitching trace of computerized sewing machines. In this paper, we will study the Hamiltonian cycle property of linear-convex supergrid graphs which form a subclass of supergrid graphs. A connected graph is called k-connected if there are k vertex-disjoint paths between every pair of vertices, and is called locally connected if the neighbors of each vertex in it form a connected subgraph. In this paper, we first show that any 2-connected, linear-convex supergrid graph is locally connected. We then give constructive proofs to show that any 2-connected, linear-convex supergrid graph contains a Hamiltonian cycle. Based on the constructive proofs, we finally present a linear-time algorithm to construct a Hamiltonian cycle of a 2-connected, linear-convex supergrid graph.

1. Introduction

A *Hamiltonian cycle* in a graph is a simple cycle in which each vertex of the graph appears exactly once. The *Hamiltonian cycle* problem involves determining whether or not a graph contains a Hamiltonian cycle. A graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle. The Hamiltonian path problem is defined similarly. They have numerous applications in different areas, including establishing transport routes, production launching, the on-line optimization of flexible manufacturing systems [1], computing the perceptual boundaries of dot patterns [29], pattern recognition [2,30,33], and DNA physical mapping [14]. It is well known that the Hamiltonian cycle and path problems are NP-complete for general graphs [10,21]. The same holds true for bipartite graphs [24], split graphs [11], circle graphs [7], undirected path graphs [3], planar bipartite graphs with maximum degree 3 [20], grid graphs [20], triangular grid graphs [12], and supergrid graphs [18].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Graph G is called *connected* if there exists a path between every pair of its vertices, and G is said to be k-connected ($k \geqslant 2$) if there are k vertex-disjoint paths between every pair of vertices in it. Obviously, if graph G is not 2-connected, then it is not Hamiltonian. However, a 2-connected graph does not imply that it is Hamiltonian. For a vertex v of G, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v. In general, we remove the subscript 'G' of $N_G(v)$ from the notation if it has no ambiguity. For a subset of vertices $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. A vertex v of G is said to be *locally connected* if $G[N(v)]$ is connected. Graph G is called

☆ A preliminary version of this paper has appeared in: 2015 The Third International Symposium on Computing and Networking (CANDAR 2015), Sapporo, Hokkaido, Japan, 2015, pp. 103–109.

E-mail address: rwhung@cyut.edu.tw.

http://dx.doi.org/10.1016/j.dam.2016.04.020
0166-218X/

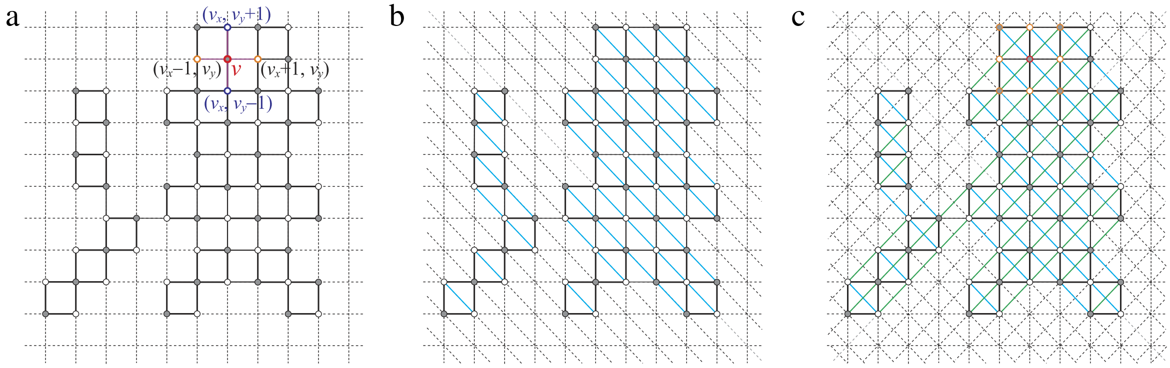

Fig. 1. (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, where circles represent the vertices and solid lines indicate the edges in the graphs.

locally connected if each vertex of G is locally connected. The locally connected property of graphs can be applied to VLSI architecture [15]. Some further survey for local connectivity can be found in [9].

The *two-dimensional integer grid* G^∞ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if and only if the (Euclidean) distance between them is equal to 1. The *two-dimensional triangular grid* T^∞ is an infinite graph obtained from G^∞ by adding all edges on the lines traced from up-left to down-right. A *grid graph* is a finite, vertex-induced subgraph of G^∞. For a node v in the plane with integer coordinates, let v_x and v_y be the *x* and *y coordinates* of node v, respectively, denoted by $v = (v_x, v_y)$. If v is a vertex in a grid graph, then its possible neighbor vertices include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, and $(v_x, v_y - 1)$. For example, Fig. 1(a) shows a grid graph. A *triangular grid graph* is a finite, vertex-induced subgraph of T^∞. If v is a vertex in a triangular grid graph, then its possible neighbor vertices include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y - 1)$, $(v_x - 1, v_y + 1)$, and $(v_x + 1, v_y - 1)$. For example, Fig. 1(b) depicts a triangular grid graph. Thus, triangular grid graphs contain grid graphs as subgraphs. Note that triangular grid graphs defined above are isomorphic to the original triangular grid graphs studied in the literature [12] but these graphs are different when considered as geometric graphs. By the same construction of triangular grid graphs from grid graphs, we have proposed a new class of graphs, namely *supergrid graphs*, in [18]. The *two-dimensional supergrid* S^∞ is an infinite graph obtained from T^∞ by adding all edges on the lines traced from up-right to down-left. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y - 1)$, $(v_x - 1, v_y + 1)$, $(v_x + 1, v_y - 1)$, $(v_x + 1, v_y + 1)$, and $(v_x - 1, v_y - 1)$. Then, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. For example, Fig. 1(c) shows a supergrid graph. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (points) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are bipartite [20] but triangular grid graphs and supergrid graphs are not bipartite.

The Hamiltonian cycle problem for grid graphs and triangular grid graphs is shown to be NP-complete [12,20]. The Hamiltonian cycle problem on supergrid graphs can be applied to control the stitching trace of a computerized sewing machine as stated in [18]. We also proved that the Hamiltonian cycle problem is NP-complete for supergrid graphs [18]. Thus, an important line of investigation is to discover the complexities of the Hamiltonian cycle problem when the input is restricted to be in special subclasses of supergrid graphs. In this paper, we will study the Hamiltonian property of a subclass of supergrid graphs, called *linear-convex supergrid graphs*. A supergrid graph G is *linearly convex* if, for every line l which contains an edge of S^∞, the intersection of l and G is either a line segment (a path in G), or a point (a vertex in G), or empty. Linear-convex triangular grid graphs are defined similarly. In general, a linear-convex supergrid graph is always a linear-convex triangular grid graph with the same vertex set, but the reverse is not true. For example, a linear-convex triangular grid graph G_t is shown in Fig. 2(a), but its corresponding supergrid graph with vertex set $V(G_t)$ is not linearly convex, as depicted in Fig. 2(b). In [12,31], the authors showed that any 2-connected, linear-convex triangular grid graph with the exception of the Star of David graph always contains a Hamiltonian cycle. However, the result cannot be applied to 2-connected, linear-convex supergrid graphs. In this paper, we first show that any 2-connected, linear-convex supergrid graph is locally connected. We then prove that any 2-connected, linear-convex supergrid graph contains a Hamiltonian cycle. Solid supergrid graphs are supergrid graphs without holds. If a supergrid graph G contains a hole, then we can find a line l such that the intersection of l and G contains at least two line segments, and, hence, G is not a linear-convex supergrid graph. Thus, every linear-convex supergrid graph is a solid supergrid graph, but the reverse is not true. For instance, Fig. 2(b) is a solid supergrid graph but is not linearly convex. Then, linear-convex supergrid graphs form a subclass of solid supergrid graphs. The complexity of the Hamiltonian cycle problem for solid supergrid graphs is still unknown. Solving the Hamiltonian cycle problem for linear-convex supergrid graphs may arise results that can help in solving the problem for solid supergrid graphs.

Related works of investigation are summarized as follows. Itai et al. [20] showed that the Hamiltonian cycle and path problems for grid graphs are NP-complete. They also gave necessary and sufficient conditions for a rectangular grid graph

Please cite this article in press as: R.-W. Hung, Hamiltonian cycles in linear-convex supergrid graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.04.020