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Dominating plane triangulations

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ABSTRACT

In 1996, Tarjan and Matheson proved that if G is a plane triangulated disc with n vertices, $\gamma(G) \le n/3$, where $\gamma(G)$ denotes the domination number of G, i.e. the cardinality of the smallest set of vertices S such that every vertex of G is either in S or adjacent to a vertex in S. Furthermore, they conjectured that the constant $1/3$ could be improved to $1/4$ for a sufficiently large n. Their conjecture remains unsettled. In the present paper, it is proved that if G is a Hamiltonian plane triangulation with n vertices and minimum degree at least 4, then $\gamma(G) \le \max\{\lceil 2n/7 \rceil, \lfloor 5n/16 \rfloor\}$. It follows immediately that if G is a 4-connected plane triangulation with n vertices, then $\gamma(G) \le \max\{\lceil 2n/7 \rceil, \lfloor 5n/16 \rfloor\}$.

1. Introduction

In 1996, Matheson and Tarjan [5] proved that if G is a plane triangulated disc then $\gamma(G) \le |V(G)|/3$. (In particular, then, the same bound on γ applies to any triangulation of the plane.) Plummer and Zha [7] proved that if G is a triangulation of the projective plane, then $\gamma(G) \le |V(G)|/3$ and if G is a triangulation of either the torus or Klein bottle, then $\gamma(G) \le \lceil |V(G)|/3 \rceil$. The latter result was sharpened by Honjo et al. [3] who showed that $\gamma(G) \le |V(G)|/3$ for graphs embedded in these two surfaces. They also showed that for any surface Σ, there is a positive integer $\rho(\Sigma)$ such that if G is embedded as a triangulation of Σ and the embedding has face-width at least $\rho(\Sigma)$, then $\gamma(G) \le |V(G)|/3$. Even more recently, Furuya and Matsumoto [2] generalized this result by showing that $\gamma(G) \le |V(G)|/3$, for every triangulation G of any closed surface.

Conjecture 1.1 (*Matheson and Tarjan, [5]*). *Let G be plane triangulation with a sufficiently large number of vertices. Then $\gamma(G) \le n/4$.*

The triangle has $\gamma(K_3) = 1 = n/3$, the octahedron shown in Fig. 1 (left) has $\gamma = 2 = n/3$ and the 7-vertex graph shown in Fig. 1 (right) has $\gamma = 2 = 2n/7 > n/4$, so this shows that one must assume $n \ge 8$ in order for the Matheson–Tarjan conjecture to be true.

More generally, in [7] the first and third authors of the present paper conjectured that if G is a triangulation of *any* non-spherical surface, then $\gamma(G) \le n/4$. Both these conjectures involving the $n/4$ bound remain unsettled.

In 2010, King and Pelsmajer [4] proved the Matheson–Tarjan's conjecture holds in the plane case when the maximum degree of the triangulation is 6. It was proved independently by Nünning [6] and by Sohn and Yuan [8] that for any graph G with n vertices and minimum degree $\delta(G) \ge 4$, $\gamma(G) \le 4n/11$.

An outerplanar graph is a graph embedded in the plane in such a way that all vertices of the graph lie on the boundary of the infinite face. An outerplanar graph is *maximal* (outerplanar) if it is not possible to add any new edge to G without

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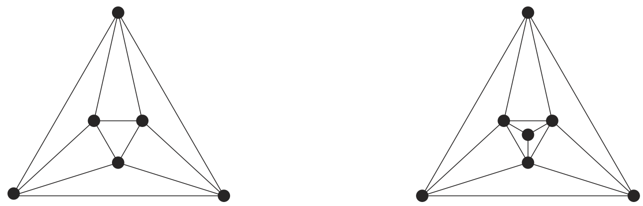

Fig. 1. Two triangulations with large domination ratio.

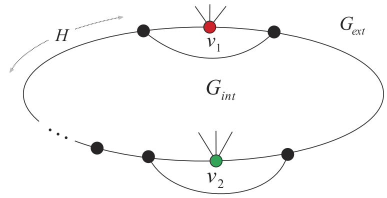

Fig. 2. A Hamilton cycle H and 2-vertices v_1 and v_2.

destroying outerplanarity. In 2013, Campos and Wakabayashi [1] proved that if G is a maximal outerplanar graph with at least $n \geq 4$ vertices, then $\gamma(G) \leq (n+t)/4$, where t is the number of vertices of degree 2 in G.

In the present paper we will show that if G is a Hamiltonian plane triangulation on n vertices with minimum degree $\delta(G) \geq 4$, then $\gamma(G) \leq \max\{\lceil 2n/7 \rceil, \lfloor 5n/16 \rfloor\}$. It follows immediately that if G is a 4-connected plane triangulation on n vertices, then $\gamma(G) \leq \max\{\lceil 2n/7 \rceil, \lfloor 5n/16 \rfloor\}$.

2. Preferred Hamilton cycles in plane triangulations

Let G be a plane triangulation with $\delta(G) \geq 4$ and suppose G contains a Hamilton cycle H. We can think of H bounding a triangulated inner subgraph G_{int} and a triangulated outer subgraph G_{ext} such that $G_{int} \cap G_{ext} = H$. Suppose $v \in V(G)$. We denote by $i\deg(v)$ (respectively, $o\deg(v)$) the degree of vertex v in G_{int} (resp. in G_{ext}).

We will need to pay particular attention to those vertices which have $i\deg(v) = 2$ or $o\deg(v) = 2$. We will call these vertices 2-*vertices*. Examples are shown in Fig. 2 where $i\deg(v_1) = 2$ and $o\deg(v_2) = 2$ respectively and hence each is an example of a 2-vertex.

Let G be a Hamiltonian plane triangulation. Such a graph may have many Hamilton cycles. We now show how to select such a cycle with certain properties we shall need later on. To this end, let H be a Hamilton cycle in G. A triangle T of G_{int} will be called *internal* (with respect to H) if $E(T) \cap E(H) = \emptyset$.

Lemma 2.1. *Let G be a plane triangulation with $\delta(G) \geq 4$ which contains a Hamilton cycle. Suppose $\gamma(G) \geq 2$. Then there exists a Hamilton cycle in G containing no three consecutive 2-vertices.*

Proof. Since G is a plane triangulation with minimum degree $\delta(G) \geq 4$, the graph $|V(G)| \geq 5$. If G has 5 vertices, then G is K_5 which is not planar. So G has at least 6 vertices. Choose a Hamilton cycle H with minimum number of 2-vertices. The lemma follows directly from the following claim.

Claim. *The Hamilton cycle H has no three consecutive 2-vertices.*

Proof of Claim. Suppose on the contrary that H contains three consecutive 2-vertices. Let three such consecutive 2-vertices be, in order, x, y and z. Let the predecessor of x be a and the successor of z be b_1. We may assume, without loss of generality, that edges $ay, yb_1 \in G_{int}$ and $xz \in G_{ext}$. Note that $a \neq b_1$ as $|V(G)| \geq 6$.

Let $P := zb_1 \cdots b_k$ be a maximal path of H such that all vertices in P are neighbors of y via an edge in G_{int}. If $b_k = a$, then all vertices of G are adjacent to y. But in this case $\gamma(G) = 1$, a contradiction to $\gamma(G) = 2$. So assume $b_k \neq a$. Since P is maximal and G is a plane triangulation, the vertex b_k has a neighbor via an edge in G_{int} other than y and the two neighbors in H.

If $k = 1$, replace the path $axyzb_1$ by $axzyb_1$ in H to obtain a new Hamilton cycle H'. Then whereas H has x, y and z as 2-vertices, H' has vertices y and z as 2-vertices. But neither x and b_1 is a 2-vertex with respect to H' since the degree of x is at least 4 and our assumption on the fourth neighbor of b_k. No other vertex changes whether or not it is a 2-vertex. (See Fig. 3.) So H' has fewer 2-vertices than does H, contradicting the assumption that H has the minimum number of 2-vertices.

In the following, suppose that $k \neq 1$. In this case, we replace the path $axyzb_1 \cdots b_{k-1}b_k$ by the path $axzb_1 \cdots b_{k-1}yb_k$ in H to obtain a new Hamilton cycle H'. Note that x and z are no longer 2-vertices of H', but b_{k-1} is a new 2-vertex of H'. No other vertex changes whether or not it is a 2-vertex. (See Fig. 4.) So H' has fewer 2-vertices than does H, a contradiction to the assumption that H has minimum number of 2-vertices. This completes the proof. □

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