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Chromatic and flow polynomials of generalized vertex join graphs and outerplanar graphs

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ABSTRACT

A generalized vertex join of a graph is obtained by joining an arbitrary multiset of its vertices to a new vertex. We present a low-order polynomial time algorithm for computing the chromatic polynomials of generalized vertex joins of trees; by duality, this algorithm can also be used to compute the flow polynomials of arbitrary outerplanar graphs. We also present closed formulas for the chromatic polynomials of generalized vertex joins of cliques, and the chromatic and flow polynomials of generalized vertex joins of cycles.

1. Introduction

Graph polynomials contain various information about the structure and properties of graphs; their study is an active area of research with many theoretical consequences and practical applications. Two of the most important single-variable graph polynomials are the chromatic and flow polynomials. Their coefficients, roots, values at specific points, and derivatives have meaningful interpretations related to the chromatic and flow numbers, Hamiltonicity [22], number of acyclic and totally cyclic orientations [19], cycle space [27], and edge-connectivity [12] of the corresponding graphs.

Chromatic and flow polynomials also have connections to other sciences such as statistical physics, combinatorics, and theoretical computer science. The chromatic polynomial is the zero-temperature limit of the anti-ferromagnetic Potts model and is used to model the behavior of crystals and ferromagnets [18]; it is also related to the Stirling and Beraha numbers, which arise in a variety of analytic and combinatorics problems (cf. [13,1]). The flow polynomial is used in crystallography and statistical mechanics to model the physical properties of ice and other crystals [11]. For more applications of chromatic and flow polynomials, see the comprehensive survey of Ellis-Monaghan and Merino [4] and the bibliography therein.

Unfortunately, computing the chromatic and flow polynomials of a graph are very challenging tasks. These problems are NP-hard for general graphs, and even for bipartite planar graphs and sparse graphs as shown in [16]. In fact, most of the terms of the chromatic and flow polynomials of general graphs cannot even be approximated (see [5,16]). Thus, a large volume of work in this area is focused on exploiting the structure of specific types of graphs in order to derive closed formulas, algorithms, or heuristics for computing their chromatic and flow polynomials. Such investigations frequently focus on classes of graphs which are generalizations of trees, cliques, and cycles.

In particular, Wakelin et al. [3,24] considered a class of graphs called polygon trees and computed their chromatic polynomials; they also characterized the chromatic polynomials of biconnected outerplanar graphs and the flow

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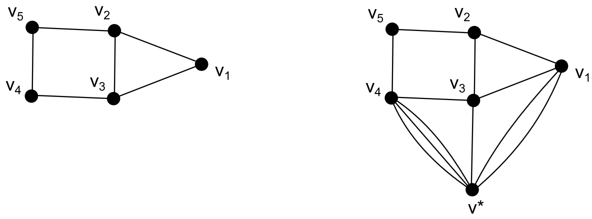

Fig. 1. *Left:* A graph G. *Right:* G_S, the generalized vertex join of G using $S = \{v_1, v_1, v_3, v_4, v_4, v_4\}$.

polynomials of their dual graphs. Whitehead [25,26] characterized the chromatic polynomials of a class of clique-like graphs called q-trees. Furthermore, Lazuka [10] obtained explicit formulas for the chromatic polynomials of cactus graphs, Gordon [6] studied Tutte polynomials (a generalization of chromatic and flow polynomials) of rooted trees, and Mphako-Banda [14,15] derived formulas for the chromatic, flow, and Tutte polynomials of flower graphs.

In this paper, we consider yet another generalization of trees, cliques, and cycles. We define a generalized vertex join of a graph G to be the graph obtained by joining an arbitrary multiset of the vertices of G to a new vertex. We compute the chromatic polynomials of generalized vertex joins of trees, cliques, and cycles, and use the duality of chromatic and flow polynomials to find the flow polynomials of certain other classes of graphs, including outerplanar graphs. Thus, we complement the work of Wakelin et al. [3,24] on chromatic polynomials of outerplanar graphs and flow polynomials of their duals, by characterizing the flow polynomials of outerplanar graphs and the chromatic polynomials of their duals. Several related results are included as well.

The paper is organized as follows. In the next section, we recall some notions and notations related to graph theory and graph polynomials. In Section 3, we list well-known technical tools used in the computation of chromatic and flow polynomials. In Section 4, we compute the chromatic polynomials of generalized vertex joins of trees; we relate these results to outerplanar graphs in Section 5. In Section 6, we consider generalized vertex joins of cliques and cycles, and related dual results. We conclude with some final remarks and open questions in Section 7.

2. Preliminaries

We assume the reader is familiar with basic graph theoretic notions and operations; refer to [2] for an extensive background on graph theory. In this section, we first recall the definition of a multiset and related terms, followed by select graph theoretic notions used in the paper.

Let $G = (V, E)$ be a graph. A *multiset* S over V is a collection of vertices of V, each of which may appear more than once in S. The number of times a vertex v appears in S is the *multiplicity* of v. The *underlying set* of S is the set S' which contains the (unique) elements of S. For example, $S = \{v_1, v_1, v_3, v_4, v_4, v_4\}$ is a multiset over $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the underlying set of S is $S' = \{v_1, v_3, v_4\}$. Using this notion, we define the *generalized vertex join of G using S* to be the graph $G_S = (V \cup \{v^*\}, E \cup \{vv^* : v \in S\})$. Note that if the multiplicity of v in S is k, there are k parallel edges between v and v^* in G_S. See Fig. 1 for an example.

Given $G = (V, E)$ and $S \subset V$, the *induced subgraph* $G[S]$ is the subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both ends in S. Given $u, v \in V$, the *contraction* G/uv is obtained by deleting edge uv if it exists, and identifying u and v into a single vertex. Note that G does not need to have the edge uv for G/uv to be defined. Finally, we say that G is *biconnected* if $G - v$ has exactly one connected component for all $v \in V$.

Many of the graphs considered in this paper are *planar* graphs — i.e., they can be drawn in the plane so that their edges do not cross each other. A graph drawn in such a way is called a *plane* graph. If G is a plane graph, its *dual* G^* is a graph that has a vertex corresponding to each face of G, and an edge joining the vertices corresponding to neighboring faces for each edge of G. Note that if G is connected, $G = (G^*)^*$. The *weak dual* of G is the subgraph of G^* whose vertices correspond to the bounded faces of G.

We close this section by introducing the two graph polynomials we will investigate in the sequel. A *vertex coloring* of G is an assignment of colors to the vertices of G so that no edge is incident to vertices of the same color. A t-coloring of G is a vertex coloring using at most t colors. The *chromatic polynomial* $P(G; t)$ counts the number of t-colorings of G; if the dependence on t is implied in the context, this can be abbreviated as $P(G)$.

A closely related polynomial is the flow polynomial. A *nowhere-zero $\mathbb{Z}_t$-flow* on G is an assignment of values from $\{1, 2, \ldots, t-1\}$ to the edges of an arbitrary orientation of G so that the total flow entering each vertex is congruent modulo t to the total flow leaving each vertex. The *flow polynomial* $F(G; t)$ counts the number of nowhere-zero $\mathbb{Z}_t$-flows on G; if the dependence on t is implied, this can be abbreviated as $F(G)$.

3. Tools for computing chromatic and flow polynomials

Before we present our main results, we list a number of well-known facts frequently used in the computation of chromatic and flow polynomials. Proofs of these and other related results are given by Tutte [23]. In what follows, let $G = (V, E)$ be a