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An algorithmic metatheorem for directed treewidth

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ABSTRACT

The notion of *directed treewidth* was introduced by Johnson et al. (2001) as a first step towards an algorithmic metatheory for digraphs. They showed that some NP-complete properties such as Hamiltonicity can be decided in polynomial time on digraphs of constant directed treewidth. Nevertheless, despite more than one decade of intensive research, the list of hard combinatorial problems that are known to be solvable in polynomial time when restricted to digraphs of constant *directed* treewidth has remained scarce. In this work we enrich this list by providing for the first time an algorithmic metatheorem connecting the monadic second order logic of graphs to directed treewidth. We show that most of the known positive algorithmic results for digraphs of constant directed treewidth can be reformulated in terms of our metatheorem. Additionally, we show how to use our metatheorem to provide polynomial time algorithms for two classes of combinatorial problems that have not yet been studied in the context of directed width measures. More precisely, for each fixed $k, w \in \mathbb{N}$, we show how to count in polynomial time on digraphs of directed treewidth w, the number of minimum spanning strong subgraphs that are the union of k directed paths, and the number of maximal subgraphs that are the union of k directed paths and satisfy a given minor closed property. To prove our metatheorem we devise two technical tools which we believe to be of independent interest. First, we introduce the notion of tree-zig-zag number of a digraph, a new directed width measure that is at most a constant times directed treewidth. Second, we introduce the notion of z-saturated tree slice language, a new formalism for the specification and manipulation of infinite sets of digraphs.

1. Introduction

Since the introduction of *directed treewidth* in [32,29] much effort has been devoted into trying to identify algorithmically useful digraph width measures. Such a width measure should ideally satisfy two properties. First, it should be small on several interesting instances of digraphs. Second, many combinatorial problems should become polynomial time tractable on digraphs of constant width. While the first property is satisfied by most of the digraph width measures introduced so far [2,6–9,24,26,27,32,34], the goal of identifying large classes of problems that can be solved in polynomial time when these measures are bounded by a constant has proven to be extremely hard to achieve. On the positive side, Johnson, Robertson, Seymour and Thomas showed already in their seminal paper [29] that certain linkage problems, such as Hamiltonicity and k-disjoint paths (for constant k), can be solved in polynomial time on digraphs of constant directed treewidth. Subsequently, It was shown in [18] that for each constant $k \in \mathbb{N}$, one can decide in polynomial time the existence of a spanning tree with at most k leaves on digraphs of constant directed treewidth. More recently, it was shown in [7] that determining the winner for some classes of parity games can be solved in polynomial time on digraphs of constant DAG-width [7].

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In this work we enrich the list of problems that can be solved in polynomial time on digraphs of constant directed treewidth. More precisely, we devise the first algorithmic metatheorem connecting *directed* treewidth to the monadic second order logic of graphs with edge set quantifications (MSO_2 logic). We show that most of the positive algorithmic results obtained so far on digraphs of constant *directed* treewidth can be reformulated in terms of our metatheorem. Additionally we show how to use our metatheorem to provide polynomial time algorithms for a parameterized version of the minimum spanning strong subgraph problem, and for a parameterized version of the problem of counting subgraphs satisfying a given minor closed property.

We note that celebrated results due to Courcelle [15] and Arnborg, Lagergren and Seese [3] state that any problem expressible in MSO_2 logic can be solved in linear time on graphs of constant *undirected* treewidth. Additionally, an equally famous result due to Courcelle, Makowsky and Rotics states that any problem expressible in MSO logic (without edge set quantifications) can be solved in linear time on graphs of constant clique-width [17]. However, we observe that there are families of digraphs of constant *directed* treewidth, but simultaneously unbounded *undirected* treewidth and clique-width [17]. For instance, the $n \times n$ grid, in which all horizontal edges are oriented to the right and all vertical edges are oriented upwards, has *directed* treewidth 0, but *undirected* treewidth $\Theta(n)$ and clique-width $\Theta(n)$. Thus our algorithmic metatheorem is not implied by the results in [15,3,17]. On the other hand, the fact that 3-colorability is MSO expressible implies that a complete analog of Courcelle's is theorem for digraphs of constant directed treewidth cannot be achieved unless P $=$ NP, since 3-colorability is already NP-complete on DAGs.

Before stating our main theorem we will introduce some notation. An edge-weighting function for a digraph $G = (V, E)$ is a function $\mu : E \to \Omega$ where Ω is a finite commutative semigroup of size polynomial in $|V|$. We will always assume that Ω has an identity element. We define the size of G as $|G| = |V| + |E|$. The weight of a subgraph $H = (V', E')$ of G is defined as $\mu(H) = \sum_{e \in E'} \mu(e)$. We say that H is the union of k directed paths if there exist directed simple paths $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ with $\mathfrak{p}_i = (V_i, E_i)$ for $i \in \{1, \ldots, k\}$ such that $H = \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_k = (\cup_{i=1}^{k} V_i, \cup_{i=1}^{k} E_i)$. We note that the unions we consider are not necessarily vertex-disjoint nor edge-disjoint.

Theorem 1 (*Main Theorem*)**.** *Let φ be an MSO_2 sentence and let $k, w \in \mathbb{N}$. There is a computable function $f(\varphi, w, k)$ such that, given a weighted digraph $G = (V, E, \mu : E \to \Omega)$ of directed treewidth w, a positive integer $l < |V|$, and an element $\alpha \in \Omega$, one can count in time $f(\varphi, w, k) \cdot |G|^{O(k \cdot (w+1))}$ the number of subgraphs H of G simultaneously satisfying the following four properties:*

(i) $H \models \varphi$,
(ii) *H is the union of k directed paths,*
(iii) *H has l vertices,*
(iv) *H has weight $\mu(H) = \alpha$.*

We note that in [20] we proved an analog theorem for digraphs of constant *directed* pathwidth. Nevertheless it can be shown that there exist families of digraphs of constant directed treewidth but unbounded *directed* pathwidth [7]. Therefore, Theorem 1 is a strict generalization of the results in [20]. To prove Theorem 1 we will introduce two new technical tools which may be of independent interest. The first, the tree-zig-zag number of a digraph, is a new directed width measure that is at most a constant times directed treewidth. The second, the notion of z-saturated tree slice languages, is a new framework for the manipulation of infinite families of digraphs.

1.1. Applications

The parameters l and α in Theorem 1 are upper bounded by $|V|^{O(1)}$. By varying these parameters we can consider different flavors of optimization problems. For instance, we can choose to count the number of subgraphs of G that are the union of k directed paths, satisfy φ and have maximal/minimal number of vertices, or maximal/minimal weight. In this section we provide a list of natural combinatorial problems that can be solved in polynomial time on digraphs of constant directed treewidth using Theorem 1. In Section 1.1.1, we show how to use Theorem 1 to rederive three known positive algorithmic results for digraphs of constant directed treewidth. In Section 1.1.2, we show how Theorem 1 can be used to solve in polynomial time two interesting classes of combinatorial problems which have not yet been studied in the context of digraph width measures. Concerning the first class of problems, we show how to count the number of *minimum spanning strong subgraphs* that are the union of k directed paths. Concerning the second class, we show how to count the number of maximal subgraphs that are the union of k directed paths and satisfy some given minor closed property.

1.1.1. First examples

In order to use Theorem 1 to solve a counting problem in polynomial time, we need to exhibit an MSO_2 sentence φ specifying a suitable class of digraphs to be counted, and to specify values for the parameters l and α which respectively determine the number of vertices and the weight of the subgraphs being counted. We observe that the class of digraphs specified by φ is fixed and does not vary with the input digraph. The parameters l and α on the other hand, may vary with the input.