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Kings and Heirs: A characterization of the (2, 2)-domination graphs of tournaments

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ABSTRACT

In 1980, Maurer coined the phrase king when describing any vertex of a tournament that could reach every other vertex in two or fewer steps. A (2, 2)-domination graph of a digraph D, $dom_{2,2}(D)$, has vertex set $V(D)$, the vertices of D, and edge uv whenever u and v each reach all other vertices of D in two or fewer steps. In this special case of the (i, j)-domination graph, we see that Maurer's theorem plays an important role in establishing which vertices form the kings that create some of the edges in $dom_{2,2}(D)$. But of even more interest is that we are able to use the theorem to determine which other vertices, when paired with a king, form an edge in $dom_{2,2}(D)$. These vertices are referred to as heirs. Using kings and heirs, we are able to completely characterize the (2, 2)-domination graphs of tournaments.

1. Introduction

Domination in digraphs has been the focus of research for decades within a variety of areas in mathematics. The current branch of research has evolved from studying dominance in animal societies in the 1950s, led by mathematical sociologist H. Landau [6–8]. Further results involving what would later be called a king in a tournament were supplied by Moon [10] in his monograph. Yet it was Maurer in 1980 [9] who coined the phrase *king* in a tournament to refer to any vertex that could beat every other vertex in at most two steps. It is that term we will use throughout this paper to refer to such a vertex, as it describes precisely the dominance we wish to explore.

Here, we are interested in a tournament, T, which is a set of n vertices where there is an arc between every pair of vertices. We say that u beats v, $u \rightarrow v$, if arc (u, v) is in T. The set of players that u beats is the outset of u, $O^+(u)$, and the set of players that beat u is the inset of u, $O^-(u)$. The distance between vertices u and v, $dist(u, v)$ or $dist_T(u, v)$, is the minimum number of arcs in a directed path from u to v.

The authors previously took the concept of (i, j) dominating sets defined by Hedetniemi et al. in their original works [5,3,4] and created (i, j)-domination graphs [1,2]. Here, we look specifically at the (2, 2)-domination graphs of tournaments. Given a digraph D, $G = dom_{2,2}(D)$ is the (2, 2)-domination graph of D where $V(G) = V(D)$ with edge uv if vertices u and v can each reach all of the remaining vertices in one or two steps. We call u and v a (2, 2)-dominating pair. The definition of $dom_{2,2}(D)$ should bring to mind the definition of a king, as any pair of kings is a (2, 2)-dominating pair. However, pairs of kings are not the only vertices to create edges in $dom_{2,2}(D)$.

For simplicity of notation we will write $T - \{x\}$ to mean the induced subtournament obtained when x is removed from the vertex set of T. Consider any (2, 2)-dominating pair, $\{u, v\}$, that creates an edge in $dom_{2,2}(D)$. Say that u beats v. Since u

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can reach all vertices, including v in one or two steps, u is a king. We know that v can reach all vertices except possibly u in one or two steps, so v must be a king in $T - \{u\}$. If v is not a king in T, then v cannot reach u in two steps, and consequently v fails to form a $(2, 2)$-dominating pair with any vertex other than u. Call such a vertex an *heir*. In other words, an heir is a vertex who is not a king, but when a particular king is removed, it becomes a king.

Lemma 1.1. *If h is an heir of king k, then h is not an heir of any other king.*

Proof. Suppose h is an heir of k_i and k_j. Then h is a king in $T - \{k_i\}$, so must beat vertex k_j in at most two steps. Thus, h is not an heir of k_j. □

In a tournament T, on n vertices with kings labeled $x_1, x_2, \ldots, x_k$ define the royal sequence as follows $[k; h_1, h_2, \ldots, h_k; r]$ with $r = n - k - \sum_{i=1}^{k} h_i$, h_i representing the number of heirs of king x_i, and r representing the number of vertices in T which are neither kings nor heirs. Note that it is not strictly necessary to provide k and r in the sequence but it is convenient to do so. Note also that we may label the kings arbitrarily so we may permute the sequence of h_i freely. In Sections 2 and 3 of this paper we will completely characterize royal sequences, and as a consequence present a complete characterization of $(2, 2)$-domination graphs of tournaments.

To create the environment in which we are working, both within the realms of kings and those of domination graphs, foundational results must be examined. First, we examine three results for kings.

Lemma 1.2 (*Landau [8]*)**.** *Any vertex with highest out degree in a tournament is a king.*

A regular tournament is one where the outdegree of every vertex is the same. Thus, every vertex in a regular tournament is a king. The next two lemmas add more information on how kings interact with vertices in the tournament.

Lemma 1.3 (*Maurer [9]*)**.** *If vertex u has nonempty inset, then u is beaten by a king.*

Corollary 1.4. *If a tournament contains exactly three kings, those kings form a three cycle.*

Now we look at the insets of u, $O^-(u)$, and outsets of u, $O^+(u)$, in relationship to subsets, which ultimately help determine which vertices are or are not kings or heirs.

Lemma 1.5 (*Factor, Langley [2]*)**.** *$u \in V(T)$ is a king if and only if for any $v \in V(T) - \{u\}$, $O^-(v) \nsubseteq O^-(u)$.*

The contrapositive to this is the following.

Corollary 1.6. *There exists a vertex $v \in V(T) - \{u\}$ where $O^-(v) \subseteq O^-(u)$ if and only if u is not a king of T.*

Since no vertex is in its own outset, we remove the equality in the subset notation and rewrite the contrapositive using the definition of a king so that it is most useful to the approach in this paper.

Corollary 1.7. *The vertex u cannot reach vertex v in two or fewer steps if and only if $O^-(v) \subset O^-(u)$ or equivalently, $O^+(u) \subset O^+(v)$.*

The next sections use some constructions requiring the union of graphs. Given two tournaments $T_1 = (V_1, A_1)$ and $T_2 = (V_2, A_2)$, then $T_1 \cup T_2$ is a directed graph with vertex set $V_1 \cup V_2$ and arcs $A_1 \cup A_2$. Since we are studying tournaments we will subsequently define arcs between all pairs of vertices $v_1 \in V_1$ and $v_2 \in V_2$ to create a tournament on $V_1 \cup V_2$.

2. Royal sequences and Maurer's theorem

In this section, we begin to examine the existence of royal sequences. We delve into the role that heirs play in ascertaining the existence of the sequences, and observe how Maurer's theorem can be used to constrain the heirs by viewing them as future kings (kings with their associated king removed). With the exception of tournaments with $k = 3$ or $k = 4$ kings, the application of Maurer's theorem and constructive lemmas allows us to determine all possible royal sequences. The cases of $k = 3$ or $k = 4$ kings require particular approaches and are reserved for Section 3.

Theorem 2.1 (*Maurer [9]*)**.** *There exists a tournament T with n vertices and k kings, $n \geq k \geq 1$, unless $k = 2$ or $k = n = 4$.*

Maurer's theorem includes all the tournaments where every vertex is a king, and thus we have the following corollaries:

Corollary 2.2. *There exists a tournament T with royal sequence $[k; 0, \ldots, 0; 0]$ if $k = 1$, $k = 3$ or $k \geq 5$.*

Corollary 2.3. *There exists no tournament T with royal sequence $[2; h_1, h_2; r]$ or $[4; 0, 0, 0, 0; 0]$.*