Discrete Applied Mathematics 204 (2016) 216–220

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Weak saturation number for multiple copies of the complete graph with a star removed☆

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ARTICLE INFO

Article history:
Received 15 January 2015
Received in revised form 19 October 2015
Accepted 21 October 2015
Available online 18 November 2015

Keywords:
Saturated graphs
Weak saturation
Multiple copies

ABSTRACT

Let K_t denote the complete graph with t vertices, and let $K_{1,m}$ (a star with m edges) denote the complete bipartite graph with partite sets of sizes 1 and m. A graph G of order n is *weakly F-saturated* if G contains no copy of F, and there is an ordering of the edges in $E(K_n \backslash G)$ so that if they are added one at a time, then each edge added creates a new copy of F. In this paper, the weak saturation number of multiple copies of $K_t - K_{1,m}$ is determined for positive integers t and m $(1 \leq m < t-1)$. This completely answers the question 3 in paper Faudree et al. (2013), partially answers the question 4 in paper Faudree et al. (2013) and the question 1 in Faudree and Gould (2014).

1. Introduction and preliminaries

Only finite graphs without loops and multiple edges will be considered. For a fixed graph G, we denote the vertex set by $V(G)$, the edge set by $E(G)$, the number of $V(G)$ by $|V(G)|$ and the number of $E(G)$ by $|E(G)|$. For positive integers m and t, K_t denotes the complete graph of order t, and $K_{n,m}$ denotes the complete bipartite graph with partite sets of sizes n and m, respectively. For $n = 1$, $K_{1,m}$ is also known as a star with m edges. For convenience, we use kH to denote a graph that has k components each of them isomorphic to H throughout this paper. For the terms, we do not define here, please refer to [1].

For a fixed graph F, a graph G is *F-saturated* if there is no copy of F in G, but for any edge $e \notin E(G)$, there is a copy of F in $G \bigcup \{e\}$. The collection of F-saturated graphs of order n is denoted by $SAT(n, F)$. The saturation number, denoted as $sat(n, F)$, is the minimum number of edges in a graph in $SAT(n, F)$. A graph G of order n is *weakly F-saturated* if G contains no copy of F, and there is an ordering of edges of $E(K_n \backslash G)$ so that if they are added one at a time, each edge added creates a new copy of F. The minimum size of a weakly F-saturated graph G of order n is denoted by $wsat(n, F)$. The set of graphs of order n that are weakly F-saturated will be denoted by $wSAT(n, F)$, and those graphs in $wSAT(n, F)$ with $wsat(n, F)$ edges will be denoted by $\underline{wSAT}(n, F)$. Clearly $wsat(n, F) \leqslant sat(n, F)$ as any F-saturated graph is also weakly F-saturated.

This study is devoted to finding the weak saturation number of certain graphs. Paper [7] considered sparse saturated graphs, while paper [3] considered saturated r-uniform hypergraphs. The investigation of weakly saturated graphs was initiated by Borowiecki and Sidorowicz [2]. In fact, a similar work has been provided by Pikhurko [8] one year earlier which is about weakly saturated hypergraphs. Later, Sidorowicz [9] considered the size of weakly saturated graphs. Recently, J. Faudree, R. Faudree and J. Schmitt [4] gave a survey of saturated graphs and more results are obtained by Faudree, Gould and Jacobson [6]. Weak saturation numbers for many families of disjoint copies of connected graphs were determined by Faudree and Gould [5]. Our research was motivated by the work of [5,6]. Paper [6] gives question 3. Is $wsat(n, K_t - K_{1,m})$

☆ Project supported by the Education Department of Henan Province, under Grant No. 14A110026.
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http://dx.doi.org/10.1016/j.dam.2015.10.028
0166-218X/

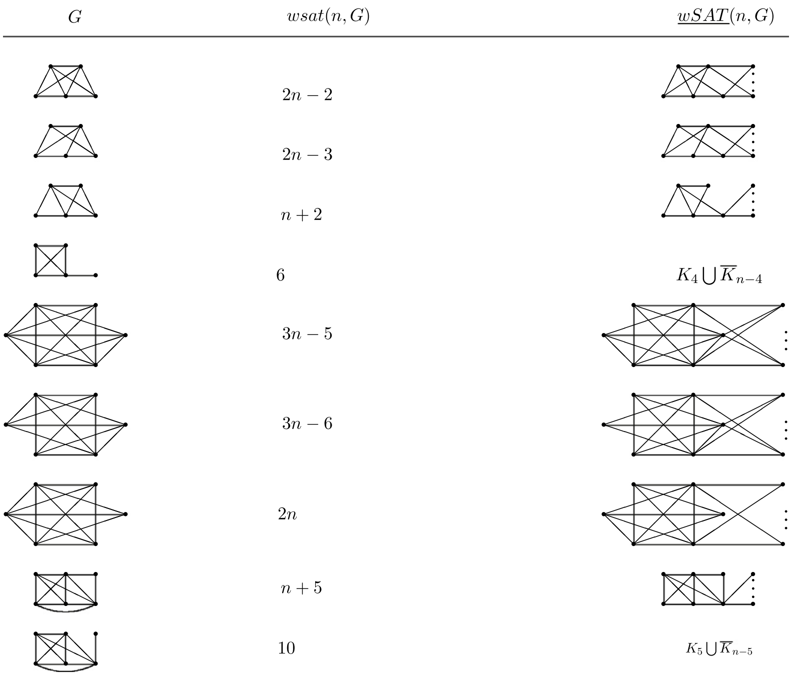

Fig. 1. Small order graphs.

$= n(t-m-2) - \frac{t^2-(2m+3)t}{2} - (m+1)$ for $2 \le m < t-1$? and question 4. Is $wsat(n, K_t - sK_2) = \binom{t-1}{2} - s + (n-t+1)(t-3)$ for $1 \le s < \frac{t-1}{2}$? Paper [5] gives $2n+k-3 \le wsat(n, k(K_5 - K_2)) \le 2n+2k-4$ and question 1. For $k \ge 2$ and n sufficiently large, for which connected graphs G, is $wsat(n, kG) = wsat(n, G) + k - 1$? In this paper, we determine $wsat(n, K_t - K_{1,m})$ which completely answers the question 3 in paper [6]. We also give $wsat(n, K_t - K_2)$ and $wsat(n, K_t - 2K_2)$ which partially answer the question 4 in paper [6]. Furthermore, we determine $wsat(n, k(K_5 - K_2))$, $wsat(n, k(K_t - K_2))$ and $wsat(n, k(K_t - K_{1,m}))$ by the technique in [5]. That partially answers the question 1 in [5].

2. Preliminary results

Fig. 1. shows the small order graphs of Propositions 1–3. Proposition 1 is the extension of Lemma 1 [6]. The techniques used in Proposition 1 are the same as that in Lemma 1.

Lemma 1 *([6]). For $n \ge 5$, $wsat(n, K_5 - K_2) = 2n - 2$.*

Proposition 1. *For $t \ge 5$ and $n \ge t$, $wsat(n, K_t - K_2) = n(t-3) - \frac{t^2-5t+4}{2}$.*

Proof. Let $L(n) = n(t-3) - \frac{t^2-5t+4}{2}$. We will show that if $G \in wSAT(n, K_t - K_2)$, then $|E(G)| \ge L(n)$. Consider the graph $G \in wSAT(n, K_t - K_2)$. When the first edge is added to G, it must create a new copy of $K_t - K_2$. This means there exists a subgraph $H_0 \subset G$, where $V(H_0) = t$, $E(H_0) = \binom{t}{2} - 2$. Since each edge of $E(K_t \backslash H_0)$ added can generate a new copy of $K_t - K_2$, H_0 and the added edges form $H_0^*(=K_t)$. If there exists a vertex $v_1 \in V(G \backslash H_0^*)$ which has at least $t-3$ neighbors in H_0^*, then each edge from v_1 to the remaining vertices in H_0^* can be added to give a new copy of $K_t - K_2$ and get $H_1^*(=K_{t+1})$. If there exists a vertex $v_2 \in G \backslash H_1^*$ which has at least $t-3$ neighbors in H_1^*, then each edge from v_2 to the remaining vertices in H_1^* can be added to get $H_2^*(=K_{t+2})$.

Either this process is repeated $n-t$ times, producing $H_{n-t}^*(=K_n)$ and $|E(G)| \ge (n-t)(t-3) + |E(H_0)| = L(n)$, or the process terminates with a subgraph $M_1^*(=K_m)$ with $m < n$, where $|E(M_1^* \bigcap G)| \ge L(m)$ and each vertex of $G - M_1^*$ will have at most $t-4$ neighbors in M_1^*. In order to give a new copy of $K_t - K_2$ as the next edge added to G, there must exist a subgraph $H_1 \subset G$, where $H_1 \not\subseteq M_1^*$ and $|V(H_1)| = t$, $|E(H_1)| = \binom{t}{2} - 2$. All the edges of $E(K_t \backslash H_1)$ can be added to form $H_1^{**}(=K_t)$. If there exists a vertex $u_1 \in G \backslash H_1^{**}$ which has at least $t-3$ neighbors in H_1^{**}, then each edge from u_1 to the remaining vertices in H_1^{**} can be added to get a new copy of $K_t - K_2$ and get $H_2^{**}(=K_{t+1})$. Continue this process until a complete graph $H_{m'-t+1}^{**}(=K_{m'})$ with $m' < n$ is obtained where any vertex $g \in G \backslash H_{m'-t+1}^{**}$ has at most $t-4$ neighbors in $H_{m'-t+1}^{**}$ (denoted by M_2^*). In order to create a new copy of $K_t - K_2$ between M_1^* and M_2^*, without loss of generality, one can assume that M_1^* and M_2^* have $t-4$ vertices in common by selecting the correct starting subgraph H_1. The two graphs M_1^* and M_2^* will contain $m + m' - (t-4)$ vertices and at most $\binom{t-4}{2}$ edges in common of the original graph G. Thus, the two graphs M_1^* and M_2^* will