Discrete Applied Mathematics 198 (2016) 29–47

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Games on concept lattices: Shapley value and core

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ARTICLE INFO

Article history:
Received 21 July 2014
Received in revised form 13 March 2015
Accepted 8 August 2015
Available online 1 September 2015

Keywords:
Cooperative game
Restricted cooperation
Concept lattice
Core
Shapley value

ABSTRACT

We introduce cooperative TU-games on concept lattices, where a concept is a pair (S, S') with S being a subset of players or objects, and S' a subset of attributes. Any such game induces a game on the set of players/objects, which appears to be a TU-game whose collection of feasible coalitions is a lattice closed under intersection, and a game on the set of attributes. We propose a Shapley value for each type of game, axiomatize it, and investigate the geometrical properties of the core (non-emptiness, boundedness, pointedness, extremal rays). In particular, we derive the equivalence of the intent and extent core for the class of distributive concepts.

1. Introduction

Cooperative games with transferable utility (TU-games) have been widely studied and used in many domains of applications. N being a set of players, or more generally, a set of abstract objects, a TU-game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ assigns to every coalition or group $S \subseteq N$ a number representing its "worth" (monetary value: benefit created by the cooperation of the members of S, or cost saved by the common usage of a service by the members of S, power, importance, etc.).

Once the function v is determined, the main concern of cooperative game theory is to provide a rational scheme for distributing the total worth $v(N)$ of the cooperation among the members of N (or determining individual power/importance degrees, if $v(N)$ is not interpreted as a monetary value). Until now most popular methods to achieve this are the Shapley value [19] and the core [15]. The Shapley value yields a single distribution vector, satisfying a set of four natural axioms (Pareto optimality, symmetry, linearity, null player property), while the core is a set of distribution vectors that are Pareto optimal and satisfy coalitional rationality (i.e., a coalition receives at least its own worth). While the Shapley value always exists for any game, the core is a convex polyhedron, but may be empty.

In many situations, however, not all subsets of N can be realized as coalitions or are feasible, which means that the mapping v is defined on a subcollection $\mathcal{F}$ of 2^N only. Pioneering works considering this situation are due to Aumann and Drèze [3], who speak of *coalition structure*, and later to Faigle and Kern [10], who speak of *restricted cooperation*. $\mathcal{F}$ has been studied under many structural assumptions, such as distributive lattices (closed under union and intersection) [13], convex geometries [5,4], antimatroids [2], union-stable systems [1] (a.k.a. weakly union-closed systems [12,11]), etc. In this case, the study of the geometric properties of the core is challenging since the core may become unbounded or have no vertices (see a survey in [16]). Also the Shapley value has to be redefined, and its axiomatization may become difficult.

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http://dx.doi.org/10.1016/j.dam.2015.08.004
0166-218X/

In many cases, the structural assumptions on $\mathcal{F}$ are not clearly motivated or are too restrictive. The aim of this paper is to study a structure for $\mathcal{F}$ which is both fairly general (a lattice of sets closed under intersection), and produced in a natural way, through a set of attributes possessed by the players or objects in N.[1] In short, our framework is based on *concept lattices* [6,7,17], a notion which has lead to the now quite active field of *formal concept analysis* [14]. M being a set of attributes, a concept is a pair (S, S') with $S \subseteq N$ and $S' \subseteq M$ such that S' is the set of those attributes that are satisfied by all members of S. A remarkable result is that any (finite) lattice is isomorphic to a concept lattice, and that the lattice of extents (i.e., the lattice of concepts (S, S') limited to the first arguments S) is a set lattice closed under intersection, and moreover any such lattice arises that way. We define a game v on the lattice of concepts, dividing it into a game v_N on the lattice of extents (which corresponds to a game with restricted cooperation $(\mathcal{F}, v)$ where $\mathcal{F}$ is a lattice closed under intersection), and a game v_M on the lattice of intents (which corresponds to a game on the set of attributes). For both types of games, we propose a Shapley value with its axiomatization. Moreover, we investigate in details the properties of the core. Our results can be seen to generalize many results of the literature on games with restricted cooperation.

The paper is organized as follows. Section 2 introduces the main notions needed in the paper: cooperative games, concept lattices and games on concept lattices. Section 3 proposes a definition for the Shapley value, which is a natural generalization of those values presented by Faigle and Kern [13], and Bilbao and Edelman [4], together with its axiomatization. Section 4 studies the properties of the core: nonemptiness, boundedness, pointedness, and extremal rays. Some interesting properties of balanced collections are also presented.

2. Framework

2.1. Cooperative games

Let $N = \{1, \ldots, n\}$ be a finite set of players. A *cooperative (TU) game* (or *game* for short) on N is a mapping $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Any subset $S \subseteq N$ is called a *coalition*. The quantity $v(S)$ represents the "worth" of the coalition, that is, depending on the application context, the benefit realized (or cost saved, etc.) by cooperation of the members of S.

We consider the general case where the cooperation is restricted, i.e., where the set $\mathcal{F}$ of all feasible coalitions might be a proper subset of 2^N. We denote the corresponding *game with restricted cooperation* as a pair $(\mathcal{F}, v)$, or simply v if there is no ambiguity.

Let us consider a cooperative game $(\mathcal{F}, v)$ with $N \in \mathcal{F}$. A payoff vector is a vector $x \in \mathbb{R}^n$. For any $S \subseteq N$, we denote by $x(S) = \sum_{i \in S} x_i$ the total payoff given by x to the coalition S. The payoff vector x is *efficient* if $x(N) = v(N)$. The core of a cooperative game is the set of efficient payoff vectors such that no coalition can achieve a better payoff by itself:

$$\mathrm{core}(\mathcal{F}, v) = \{x \in \mathbb{R}^n \mid x(S) \geq v(S)\ \forall S \in \mathcal{F},\ \text{and}\ x(N) = v(N)\}.$$

Note that core$(\mathcal{F}, v)$ is a convex closed bounded polyhedron when $\mathcal{F} = 2^N$. In other cases, the core may be unbounded or non pointed, and its study becomes difficult (see [10] and a survey in [16]). We recall from the theory of polyhedra that a polyhedron defined by a set of inequalities $\mathbf{Ax} \geq \mathbf{b}$ is the Minkowski sum of its convex part and its conic part (the so-called recession cone), the latter being determined by the inequalities $\mathbf{Ax} \geq \mathbf{0}$, and being therefore independent of the right hand side $\mathbf{b}$. So the recession cone of core$(\mathcal{F}, v)$ is the polyhedron core$(\mathcal{F}, 0)$, which does not depend on v.

A collection $\mathcal{B} \subseteq \mathcal{F}$ of nonempty sets is said to be *balanced* if there exist positive weights λ_S, $S \in \mathcal{B}$ such that

$$\sum_{S \in \mathcal{B}, S \ni i} \lambda_S = 1 \quad \forall i \in N.$$

A game $(\mathcal{F}, v)$ is said to be *balanced* if $v(N) \geq \sum_{S \in \mathcal{B}} \lambda_S v(S)$ holds for every balanced collection $\mathcal{B}$ with weight system $(\lambda_S)_{S \in \mathcal{B}}$. It is well-known that the core of v is nonempty if and only if v is balanced [10].

2.2. Concept lattices

We begin by recalling that a *lattice* is a partially ordered set (poset) $(L, \preceq)$, where $\preceq$ is reflexive, antisymmetric and transitive, such that for any two elements $x, y \in L$, a supremum $x \vee y$ and an infimum $x \wedge y$ exist. If no ambiguity occurs, the lattice is simply denoted by L. The *dual partial order* $\preceq^\partial$ is defined by $x \preceq^\partial y$ if and only if $y \preceq x$. The *dual* of the lattice $(L, \preceq)$ is the poset $(L, \preceq^\partial)$, denoted by L^∂ if no ambiguity occurs.

A *context* (see, e.g., [6,7,14,17]) is a triple $\mathcal{C} = (N, M, I)$, where N is a finite nonempty set of objects, M is a finite set of attributes, and $I : N \times M \to \{0, 1\}$ is a binary relation defined by $I(i, a) = 1$ if object $i \in N$ satisfies attribute $a \in M$, and 0 otherwise. The binary relation can be represented as a matrix or table called the *incidence matrix (table)*.

Let $\mathcal{C} = (N, M, I)$ be a context. The *intent* of a subset of objects $S \subseteq N$ is defined as the set of attributes satisfied by all objects in S:

$$S'_{\mathcal{C}} = \{a \in M \mid I(i, a) = 1,\ \forall i \in S\}.$$

[1] We do not claim for full generality, since there remain important cases which are not covered by our model. For instance, games on communication networks introduced by Myerson [18] are defined on the set of connected subsets, which are not closed under intersection in general.