Discrete Applied Mathematics 198 (2016) 48–64

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Nim with one or two dynamic restrictions

Wen An Liu *, Xiao Zhao

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, People's Republic of China

ARTICLE INFO

Article history:
Received 3 September 2014
Received in revised form 13 June 2015
Accepted 8 August 2015
Available online 28 August 2015

Keywords:
Impartial combinatorial game
Nim
Muller twist
P-position

ABSTRACT

Two classes of new games are investigated. *Nim with Pointer Restriction* is introduced by putting a pointer restriction on Nim, this pointer designates the pile that the next move must be made from. *Nim with Pointer and Modular Restrictions* is defined by putting simultaneously a pointer restriction and a modular restriction on Nim, the pointer designates the pile that the next move must be made from, the modular restriction designates the number of tokens that next move can remove from the designated pile.

The games defined above are of the form 'comply/constrain' or 'Muller twist'. The sets of all *P*-positions of these games are given by explicit formulae for any $m \geq 1$ piles.

1. Introduction

By game we mean a combinatorial game; we restrict our attention to classical impartial games. There are two conventions: in **normal play convention**, the player first unable to move is the loser (his opponent the winner); in **misère play convention**, the player first unable to move is the winner (his opponent the loser).

It has become traditional in combinatorial game theory to classify positions of impartial games into two categories: *N*-positions and *P*-positions. These two categories stand for next player wins (*N*-positions) and previous player wins (*P*-positions). The way to think of this dichotomy is that when it is your move, if the position is an *N*-position, then by optimal play you can win; if the position is a *P*-position, then no matter what move you make, when your opponent plays optimally you must lose. The theory of such games can be found in [3,5,9,11,14,6].

Proposition 1 ([7], *Characterization of the P-positions of an impartial acyclic game*)**.** *The sets of $\mathcal{P}$ and $\mathcal{N}$ positions of any impartial acyclic game (like Wythoff's game) are uniquely determined by the following two properties:*

- *Any move from a P-position leads to an N-position (stability property of the P-positions).*
- *From any N-position, there exists a move leading to a P-position (absorbing property of the P-positions).*

Proof. See Proposition 1 in [7]. □

1.1. Nim

The game of Nim is well known, see [4,10,8,17]. The game is played with piles of tokens. The two players take turns removing any positive number of tokens from any one pile. Under normal play convention, Bouton's analysis of Nim showed that the *P*-positions are those for which nim-addition (bitwise addition without carry) on the sizes of the piles is 0, and the *N*-positions are those for which nim-addition on the sizes of the piles is greater than 0. In the same paper, all *P*-positions of Nim were determined under misère play convention.

* Corresponding author.
E-mail address: liuwenan@126.com (W.A. Liu).

http://dx.doi.org/10.1016/j.dam.2015.08.006

1.2. Restriction of Nim

In Nim, two types of restrictions play an important role:
(R1) Which pile will a player remove tokens from at his turn?
(R2) How many tokens will a player remove at his turn?

If a restriction remains unchanged throughout the game, we call it a **static restriction**. For example, in Small Nim, two players are required to remove tokens from the smallest (or any equal smallest) pile. This is equivalent to putting a static restriction on (R1). Such static restrictions are adopted in Large Nim [2] and End-Nim [1].

Bounded Nim [15] is equivalent to putting a static restriction on (R2): in one-pile Nim, a pile of n tokens and a constant k are given. Two players alternately take from 1 up to k tokens from the pile.

If a restriction may vary during the game, we call it a **dynamic restriction**. In Dynamic One-Pile Nim [13], the maximum number of tokens that can be removed on each successive move changes during the play of the game.

• *Muller Twist.* F. Smith and P. Stănică [16] introduced a class of comply/constrain games (or games with Muller Twist): given m piles of tokens, a position v consists of a "physical" position $(a_1, a_2, \ldots, a_m)$ and a restricted set R, i.e., $v = (a_1, a_2, \ldots, a_m; R)$. The restricted set R given by a player indicates the number of tokens that the next player can remove.

For example, in Odd-or-Even Nim [16], the first player specifies whether the second player is to take an odd or an even number of tokens. Each move after that consists of a comply phase and a constrain phase. In the comply phase, the player gets to choose the pile but must comply with the odd or even restriction on the number of tokens removed. In the constrain phase, the player specifies that the opponent must take an odd or even positive number of tokens on his next move. In other words, let $\mathbb{Z}_{>0}^{odd} = \{x > 0 \mid x \text{ is odd}\}$ and $\mathbb{Z}_{>0}^{even} = \{x > 0 \mid x \text{ is even}\}$, then the restricted set $R = \mathbb{Z}_{>0}^{odd}$ or $\mathbb{Z}_{>0}^{even}$.

In [12], the authors generalized "odd-or-even restriction" to "k-blocking modular restrictions": given two integers n and k with $1 \leq k < n$, a constraint consists of k different subconstraints of the kind "you must not take an integer equivalent to x modulo n", $x \in \{1, 2, \ldots, n\}$. This is equivalent to putting a dynamic restriction on (R2).

• *Pointer.* In [2], M.H. Albert and R.J. Nowakowski gave another type of dynamic restriction: in Pointed Nim, the usual piles of tokens are augmented with a pointer which points to a pile even at the beginning of the game. This pointer designates the pile that the next move must be made from. To make a move a player removes one or more tokens from the designated pile, and then must move the pointer to a different, non-empty, pile (if possible), where we forbid the placement of the pointer on an empty pile when there are still non-empty piles available. In [2], the authors gave the set of all P-positions of this game. This is equivalent to putting a dynamic pointer on (R1).

If the pointer is allowed to designate the same pile as its previous placement, how can we determine the set of all P-positions of this game? This question will be answered in Section 2.

1.3. Our games and results

Definition 2. We define **Nim with Pointer Restriction** (denoted by β_m): given m piles of tokens $(a_1, a_2, \ldots, a_m)$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_m$. Two players take turns. At the beginning of the game, a pointer points to a pile. This pointer designates the pile that the next move must be made from. To make a move, a player removes any integer number of tokens from the designated pile, and then reset the pointer to an non-empty pile (the pointer is allowed to designate the same pile as its previous placement). The game ends as soon as one of the two players cannot move, and this player loses.

Example 1. If your opponent hands you a position with 3 piles of tokens (2, 3, 4) and the pointer indicates the pile of size 3, i.e., $(2, 3\downarrow, 4)$. You must remove a positive number of tokens from the pile of size 3, and reset the pointer to an non-empty pile (the pointer is allowed to designate the same pile as its previous placement).

If you remove 3 tokens, then you can hand your opponent a position $(2\downarrow, 4)$ or $(2, 4\downarrow)$; If you remove 2 tokens, then you can hand your opponent a position $(1\downarrow, 2, 4)$, or $(1, 2\downarrow, 4)$, or $(1, 2, 4\downarrow)$; If you remove 1 token, then you can hand your opponent a position $(2\downarrow, 2, 4)$ or $(2, 2, 4\downarrow)$.

Suppose that you now hand your opponent the position $(2, 4\downarrow)$. If your opponent removes all tokens from the pile of size 4 and hands you the position $(2\downarrow)$, you can remove all tokens and win. If your opponent removes 3 tokens from the pile of size 4 and hands you the position $(1\downarrow, 2)$, you must remove one token and hand your opponent the position $(2\downarrow)$. Thus your opponent wins by removing all tokens. □

Remark 1. Small Nim is equivalent to letting the pointer always be directed at a smallest pile. Similarly, Large Nim is equivalent to letting the pointer always be directed at a largest pile.

Definition 3. (1) Given an integer $n \geq 2$. For any $x \in \{1, 2, \ldots, n\}$, by $M(n; x)$ we denote the set of all positive integers equivalent to x modulo n, i.e.,

$$M(n; x) = \cup_{q=0}^{\infty}\{qn + x\}, \tag{1}$$

and by

$$\overline{M}(n; x) = \{1, 2, \ldots\} \setminus M(n; x) \tag{2}$$