Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Oriented coloring in planar, bipartite, bounded degree 3 acyclic oriented graphs

H. Coelho [a,*], L. Faria [b], S. Gravier [c], S. Klein [d]

[a] INF, UFG, and COPPE-Sistemas, UFRJ, Goiânia, GO, Brazil
[b] DCC, UERJ, Rio de Janeiro, RJ, Brazil
[c] Institut Fourier, Maths à Modeler team, CNRS - UJF, St Martin d'Hères, France
[d] IM and COPPE-Sistemas, UFRJ, Rio de Janeiro, RJ, Brazil

ARTICLE INFO

Article history:
Received 14 August 2013
Received in revised form 5 June 2015
Accepted 21 June 2015
Available online xxxx

Keywords:
Oriented coloring
Maximum degree 3 graph
Planar graph
Bipartite graph
NP-complete

ABSTRACT

It has been a challenging problem to determine the smallest graph class where a problem is proved to be hard. In the literature, this has been pointed out to be very important in order to establish the real nature of a combinatorial problem.

An *oriented k-coloring* of an oriented graph $\vec{G} = (V, \vec{E})$ is a partition of V into k subsets such that there are no two adjacent vertices belonging to the same subset, and all the arcs between a pair of subsets have the same orientation. The decision problem k − ORIENTED CHROMATIC NUMBER (OCN$_k$) consists of an oriented graph $\vec{G}$ and an integer $k > 0$, plus the question if there exists an oriented k-coloring of $\vec{G}$. By its strong appeal, many papers have presented NP-completeness proofs for OCN$_k$. It was not known the complexity status of OCN$_k$ when the input graph $\vec{G}$ satisfies that the underlying graph G has maximum degree 3.

In this paper we prove that OCN$_4$ is NP-complete for an acyclic oriented graph $\vec{G}$ such that G is at same time: connected, planar, bipartite, and with maximum degree 3.

Our result defines a P versus NP-complete dichotomy with respect to the maximum degree $\Delta(G)$: OCN$_k$ is polynomial if $\Delta(G) < 3$ and NP-complete if $\Delta(G) \geq 3$, since it is known that OCN$_3$ is in P, and that OCN$_k$ is in P when the underlying graph has $\Delta(G) \leq 2$.

1. Introduction

A natural question when studying the complexity of a graph-theoretical decision problem is to determine for which special graph classes and upper bounds on the vertex degrees the problem remains NP-complete. Given a graph $G = (V, E)$, the *orientation* of an edge $e = \{u, v\} \in E$ is one of the two possible ordered pairs uv or vu called *arcs*. An *oriented graph* $\vec{G} = (V, \vec{E})$ is obtained from G by orienting each edge of E, and $\vec{G}$ is called an *orientation* of G. If $uv \in \vec{E}$, then we say that u and v are *adjacent*. If G is connected we say that $\vec{G}$ is connected (the same for planar, and bipartite). The maximum degree of G is denoted by $\Delta(G)$ and we define $\Delta(\vec{G}) = \Delta(G)$. A *digraph* $D = (V, A)$ is a pair, where V is a set of *vertices*, and A is a set of ordered pairs of distinct elements of V, called *arcs*. A *tournament* $\vec{T}_n$ with n vertices is an orientation of the complete graph K_n. An oriented k-coloring of an oriented graph is a partition $(V_1, V_2, V_3, \ldots, V_k)$ of V into k subsets such that there are no two adjacent vertices belonging to the same subset, and all the arcs between a pair of subsets have the same orientation. The k-ORIENTED CHROMATIC NUMBER (OCN$_k$) was introduced by Courcelle [5] and then studied by Raspaud and Sopena [9].

* Corresponding author.
E-mail addresses: hebert@inf.ufg.br (H. Coelho), luerbio@cos.ufrj.br (L. Faria), sylvain.gravier@ujf-grenoble.fr (S. Gravier), sula@cos.ufrj.br (S. Klein).

http://dx.doi.org/10.1016/j.dam.2015.06.023

OCN$_k$-k-ORIENTED CHROMATIC NUMBER

INSTANCE: Oriented graph $\vec{G} = (V, \vec{E})$ and a positive integer k.

QUESTION: Is there an oriented k-coloring of $\vec{G}$?

The *oriented chromatic number* $\chi_o(\vec{G})$ of an oriented graph $\vec{G}$ is the smallest k such that $\vec{G}$ admits an oriented k-coloring. Let $\vec{G_1}$ and $\vec{G_2}$ be two oriented graphs, a *homomorphism* of $\vec{G_1}$ to $\vec{G_2}$ is a mapping $f: V(G_1) \to V(G_2)$ such that $f(u)f(v) \in \vec{E}(\vec{G_2})$, whenever $uv \in \vec{E}(\vec{G_1})$. In this case, we say that $\vec{G_1}$ is $\vec{G_2}$*-colorable*, that the vertices of $\vec{G_2}$ are the *colors* assigned to the vertices of $\vec{G_1}$, and that $\vec{G_2}$ is the *color digraph* of $\vec{G_1}$. Clearly, an oriented graph $\vec{G}$ has an oriented k-coloring if and only if there is a tournament $\vec{T_k}$ with k vertices, such that $\vec{G}$ has a homomorphism to $\vec{T_k}$.

In the fundamental paper, Bang-Jensen, Hell and MacGillivray [1] proved that if $\vec{T}$ is a tournament with at least two directed cycles, then it is NP-complete to decide whether a digraph $\vec{G}$ has an homomorphism to $\vec{T}$. Klostermeyer and MacGillivray [8] proved in 2004 using [1], that OCN$_4$ is NP-complete. They established a P versus NP-complete dichotomy with respect to the number of colors k: OCN$_k$ is polynomial if $k \leq 3$ and NP-complete if $k > 3$.

In 1978, Yannakakis [15] pointed out that from the algorithmic point of view it is important to determine the best possible bounds on the node-degree and constrained classes for which a decision problem remains NP-complete. Ries and de Werra studied in [10,11] coloring problems on mixed graphs and proved the complexity status of such problems in cubic, planar bipartite graphs.

By its strong appeal, OCN$_k$ complexity has been exhaustively studied. Two recent papers have presented proofs for NP-completeness [6,7] both using the NP-complete problem 3-SAT, each one trying to add some improvements in the previous results. In 2006, Culus and Demange [6] presented two results: that OCN$_4$ is NP-complete on acyclic oriented graphs with maximum degree $\Delta(G) = \max(p + 3; 6)$, and that OCN$_4$ is NP-complete on bipartite oriented graphs with maximum degree $\Delta(G) = \max(p + 3; 7)$, where p denotes the maximum number of occurrences of a literal. Most recently, in 2010, Ganian and Hliněný [7] got an improvement in the Culus and Demange acyclic result proving that OCN$_4$ is NP-complete for connected acyclic oriented graphs with maximum degree $\Delta(G) = \max(p + 2; 4)$. We notice that the complexity status of OCN$_k$ is not known for oriented graphs with maximum degree $\Delta(G) = 3$.

In the present work, we prove that OCN$_k$ remains NP-complete even when restricted to a connected, planar, bipartite and acyclic oriented graph $\vec{G}$ with maximum degree $\Delta(G) = 3$. Our result also establishes a P versus NP-complete dichotomy of OCN$_k$ with respect to the maximum degree $\Delta(G)$, since when $\Delta(G) \leq 2$, we know that OCN$_k$ is a polynomial problem [12]. Hence, OCN$_k$ is NP-complete if $\Delta(G) \geq 3$ and polynomial if $\Delta(G) \leq 2$. This NP-completeness result is obtained using the NP-complete problem [2,3]:

P3SAT$_{\overline{3}}$—PLANAR 3SAT WITH AT MOST 3 OCCURRENCES PER VARIABLE

INSTANCE: Set U of variables and collection C of clauses over U, $|U| = n$ and $|C| = m$, such that: (i) each clause $c \in C$ satisfies $|c| = 3$ or $|c| = 2$; (ii) each variable has 3 or 2 occurrences and each negative literal occurs once in C; (iii) the bipartite graph $G = (V, E)$ is planar and connected, where $V = U \cup C$ and E contains the pairs (u, c) if and only if either u or $\overline{u}$ belongs to clause c.

QUESTION: Is there a satisfying truth assignment for U satisfying all clauses of C?

An extended abstract was published [4] in: *Proceedings of the VII Latin-American Algorithms, Graphs and Optimization Symposium, LAGOS'2013.*

In the next section we construct the OCN$_k$ instance $[\vec{G} = (V, \vec{E}), k]$ defined from $I = (U, C)$ a P3SAT$_{\overline{3}}$ instance.

2. The special instance $[\vec{G} = (V, \vec{E}), k]$ of OCN$_k$

Let $\vec{G} = (V, \vec{E})$ be an oriented graph and ϕ an oriented coloring of $\vec{G}$. We denote the vertices of $\vec{G}$ by lower case letters and the color of ϕ by capital letters. Sometimes we depict a list of colors split by a bar "|" to represent the colors allowed for a vertex.

We use the component design technique. For this purpose, from an instance $I = (U, C)$ of P3SAT$_{\overline{3}}$, we construct for each variable u_i of U a *truth setting* $\vec{T_i}$ and for each clause c_j of C a *satisfaction testing* $\vec{S_j}$.

Our truth Setting $\vec{T_i}$ is made up from some copies of an oriented graph which we called *jellyfish* $\vec{J}_i^d$ described by the vertex set $V(\vec{J}_i^d) = \{i_i^d, j_i^d, k_i^d, \ell_i^d, m_i^d, n_i^d, o_i^d, p_i^d, q_i^d, r_i^d, s_i^d, t_i^d, u_i^d, v_i^d, x_i^d\}$ and $\vec{E}(\vec{J}_i^d) = \{i_i^d k_i^d, j_i^d i_i^d, k_i^d n_i^d, j_i^d \ell_i^d, m_i^d \ell_i^d, n_i^d m_i^d, \ell_i^d o_i^d, m_i^d p_i^d, n_i^d q_i^d, o_i^d p_i^d, q_i^d p_i^d, r_i^d o_i^d, q_i^d r_i^d, r_i^d s_i^d, s_i^d t_i^d, t_i^d u_i^d, u_i^d v_i^d, v_i^d x_i^d\}$, $i \in \{1, 2, \ldots, n\}$, $d \in \{1, 2, 3\}$. See Fig. 1(a).

Lemma 2.1. *If $\vec{G} = (V, \vec{E})$ contains $\vec{J}_i^d$ as a subgraph, $\chi_o(\vec{G}) \leq 4$ and the colors A, T, B, F (not necessarily distinct) are respectively assigned to vertices $o_i^d, p_i^d, q_i^d, r_i^d$, then:*

Please cite this article in press as: H. Coelho, et al., Oriented coloring in planar, bipartite, bounded degree 3 acyclic oriented graphs, Discrete Applied Mathematics (2015), http://dx.doi.org/10.1016/j.dam.2015.06.023