Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On quasi-monotonous graphs

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ARTICLE INFO

Article history:
Received 19 February 2013
Received in revised form 27 May 2015
Accepted 1 June 2015
Available online xxxx

Keywords:
b-coloring
Chordal graphs
Claw-free graphs

ABSTRACT

A *dominating coloring* by k colors is a proper k coloring where every color i has a representative vertex x_i adjacent to at least one vertex in each of the other classes. The *b-chromatic number*, $b(G)$, of a graph G is the largest integer k such that G admits a dominating coloring by k colors. A graph $G = (V, E)$ is said *b-monotonous* if $b(H_1) \geq b(H_2)$ for every induced subgraph H_1 of G and every subgraph H_2 of H_1. Here we say that a graph G is *quasi b-monotonous*, or simply quasi-monotonous, if for every vertex $v \in V$, $b(G - v) \leq b(G) + 1$. We shall study the quasi-monotonicity of several classes. We show in particular that chordal graphs are not quasi-monotonous in general, whereas chordal graphs with large b-chromatic number, and $(P, coP, chair, diamond)$-free graphs are quasi-monotonous. The $(P_5, P, dart)$-free graphs are monotonous. Finally we give new bounds for the b-chromatic number of any vertex deleted subgraph of a chordal graph.

1. Introduction

All graphs considered here are simple and undirected. We denote by P_n (respectively C_n) an elementary path (resp. an elementary cycle) with n vertices. Let G be a graph with a proper coloring. Let $\omega(G)$ be the maximum size of a clique of G, and $\chi(G)$ be the chromatic number of G. $\omega(G)$ is called the *clique number of* G. For any two disjoint subsets A and B, let $|A|$ be the cardinality of the set A; let $E(A, B)$ be the set of edges of G with one extremity in A and the other in B, and let $e(A, B)$ be the cardinality of $E(A, B)$. Let $\langle A \rangle$ be the subgraph of G induced by A. If H is any graph, we say that G is H*-free* if G contains no induced subgraph isomorphic to H. Let u_i be any vertex u of color i. Let us denote by $\mathcal{C}_i$ the set of vertices of color i, this set is called the class of color i. If y is a vertex of the graph G, let $N_i(y)$ be the set of neighbors of y of color i; while for any non-zero integer p, $N^p(y)$ is the set of vertices at distance exactly p from y.

A *chordal* graph is a graph where every cycle of length at least 4 has at least one chord. A *claw-free* graph is a graph such that for every vertex v the neighborhood $N(v)$ contains no three independent vertices. A *diamond* is the complete graph K_4 minus an edge. A *dart* is a diamond with a pendant vertex u adjacent to a vertex a of degree 3 of the diamond. A P_4*-sparse graph* is a graph where every 5-vertex subset contains at most one induced P_4.

A vertex v is a partner of $A \subset V$ if $A \cup v$ has at least two induced P_4. A P_4*-tidy* graph is such that every induced P_4 has at most one partner.

In a proper coloring, a vertex x_i of color i is said a *dominating vertex* if x_i is adjacent to at least one vertex in each of the other classes. The vertex x_i is also called a dominant. The color i is said *dominating* if there exists at least a vertex of color i which is dominating. A *dominating coloring* or *b-coloring* by k colors is a proper k coloring where every color i has at least a dominating vertex. The *b-chromatic number*, $b(G)$, of a graph G is the largest integer k such that G admits a dominating coloring by k colors. A dominating coloring with $b(G)$ colors will be also called an optimal b-coloring.

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http://dx.doi.org/10.1016/j.dam.2015.06.001

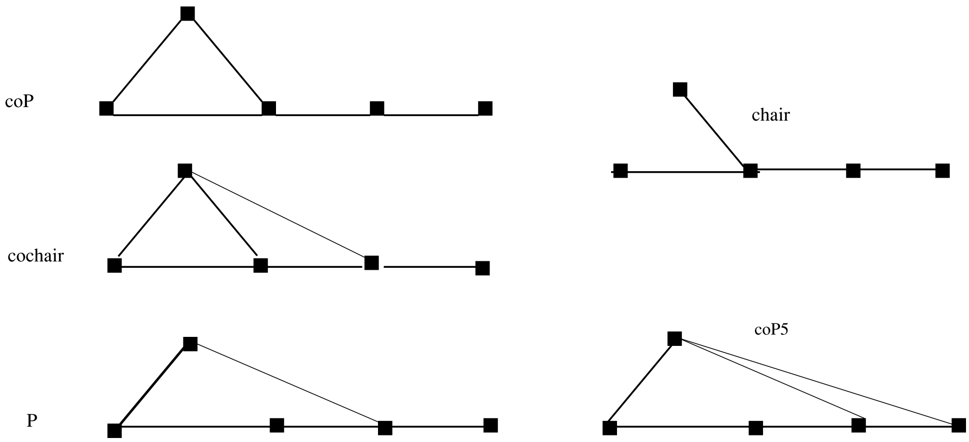

Fig. 1. Extensions of P_4.

This parameter was defined by Irving and Manlove [14]. They proved that determining $b(G)$ for an arbitrary graph G is an NP-complete problem, even if G is bipartite [17], whereas it is polynomial for trees. The problem of approximating this parameter has been studied in [6,10].

For a given graph G, it may be easily remarked that $\chi(G) \leq b(G) \leq \Delta(G) + 1$. If we are limited to regular graphs, Kratochvil et al. proved in [17] that for a d-regular graph G with at least d^4 vertices, $b(G) = d + 1$, result extended, by [5] and [9], to graphs with at least $2d^3$ vertices. Kouider and El Sahili [8] proved that for every regular graph of girth 5 and no induced cycle C_6 the same equality holds.

Let v be any vertex of a graph G. It is known that $\chi(G-v) \leq \chi(G)$. The function χ is said monotonous. This is not the case for the b chromatic number. A graph $G = (V, E)$ is called *b-monotonous* if $b(H_1) \geq b(H_2)$ for every induced subgraph H_1 of G and every subgraph H_2 of H_1. This was a definition of Bonomo et al. [1]. Here we say that a graph G is *quasi b-monotonous*, or simply quasi-monotonous, if for every vertex $v \in V$, $b(G - v) \leq b(G) + 1$. It was shown by Bonomo et al. [1] that P_4-sparse graphs are b-monotonous, later this result was extended to P_4-tidy graphs in [2]. On the other hand, we showed in [16] that every graph of girth at least 5 is b-monotonous.

A graph G is said b-perfect if $b(H) = \chi(H)$ for every induced subgraph of G (see [12]). Note that b-perfect graphs are b-monotonous. Recently, Hoang et al. have characterized the b-perfect graphs by forbidden subgraphs [13]. For other results on b-coloring one can see [10,7,12] [15] or [9].

Extensions of the class of P_4-free graphs. The numerous structural and algorithmic properties of P_4-free graphs lead researchers to define and work on extensions of this class. Many problems of optimization which are NP-Hard can be solved efficiently on those classes. Some of the classes, as the class of P_4-tidy graphs, are defined by conditions on extensions of P_4 in the graph, or by a list of forbidden subgraphs on 5 vertices, among them $P_5, coP_5, P, chair, cochair, coP$ (see Fig. 1). For many such classes, the decomposition of these graphs has been established (see for example [3,4]). The class of $(P_5, coP_5, cochair)$-free graphs was defined, by Giakoumakis and Fouquet, as the class of semi-P_4-sparse graphs, a superclass of P_4-sparse graphs. The class $(P, coP, chair, cochair)$-free deserves also the name of semi P_4-sparse. The decomposition of the semi-P_4-sparse graphs is done in [3] and [11]. $(P, coP, chair, diamond)$-free graphs are a subclass of semi-P_4-sparse graphs. For the class of $(P_5, P, cochair)$-free graphs one can see [3].

For general graphs S.F. Raj and R. Balakrishnan proved that

Theorem 1 *([18]). For every connected graph of order $n \geq 5$, and for every vertex $v \in V(G)$,*

$$b(G) - (\lceil n/2 \rceil - 2) \leq b(G - v) \leq b(G) + (\lfloor n/2 \rfloor - 2).$$

Furthermore they determined the extremal graphs. For example, in the bipartite graph $K_{p,p}$ minus $p - 1$ edges of a matching the deletion of a vertex of degree p increases the b-chromatic number by $p - 2$.

Now we give our results. In part A, we examine quasi-monotonicity of some classes. In part B we show that chordal graphs are not quasi-monotonous in general, and we give upper bounds for the b-chromatic number of their respective vertex-deleted graphs. Furthermore we show that chordal graphs of high b-chromatic number are quasi-monotonous.

(A) **Quasi-Monotonicity in some classes of graphs**

2. Results

Theorem 2. *Let $G = (V, E)$ be a graph.*

(1) *If each vertex is contained in at most two cycles of length 4, then G is quasi-b-monotonous.*

(2) *If G is $(P, coP, Chair, diamond)$-free, then G is quasi-b-monotonous.*

Please cite this article in press as: M. Kouider, On quasi-monotonous graphs, Discrete Applied Mathematics (2015), http://dx.doi.org/10.1016/j.dam.2015.06.001