ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The simultaneous metric dimension of graph families

Yunior Ramírez-Cruz[a], Ortrud R. Oellermann[b], Juan A. Rodríguez-Velázquez[a,*]

[a] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain

[b] Department of Mathematics and Statistics, University of Winnipeg, Winnipeg, MB R3B 2E9, Canada

A R T I C L E I N F O

Article history:
Received 24 December 2014
Received in revised form 23 May 2015
Accepted 15 June 2015
Available online xxxx

Keywords:
Metric dimension
Simultaneous metric dimension
Simultaneous metric generator
NP-hardness
Families of trees
Bounds

A B S T R A C T

Let $G = (V, E)$ be a connected graph. A vertex $v \in V$ is said to *resolve* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subseteq V$ is said to be a *metric generator* for G if any pair of vertices of G is resolved by some element of S. A minimum cardinality metric generator is called a *metric basis*, and its cardinality, $\dim(G)$, the *metric dimension* of G. A set $S \subseteq V$ is said to be a simultaneous metric generator for a graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$, defined on a common (labeled) vertex set, if it is a metric generator for every graph of the family. A minimum cardinality simultaneous metric generator is called a simultaneous metric basis, and its cardinality the simultaneous metric dimension of $\mathcal{G}$. We obtain sharp bounds for these invariants for general families of graphs and calculate closed formulae or tight bounds for the simultaneous metric dimension of several specific graph families. For a given graph G we describe a process for obtaining a lower bound on the maximum number of graphs in a family containing G that has simultaneous metric dimension equal to $\dim(G)$. It is shown that the problem of finding the simultaneous metric dimension of families of trees is *NP*-hard. Sharp upper bounds for the simultaneous metric dimension of trees are established. The problem of finding this invariant for families of trees that can be obtained from an initial tree by a sequence of successive edge-exchanges is considered. For such families of trees sharp upper and lower bounds for the simultaneous metric dimension are established.

1. Introduction

A generator of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of S. Given a simple and connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{N} \cup \{0\}$, where $d_G(x, y)$ is the length of a shortest path between u and v and $\mathbb{N}$ is the set of positive integers. Then (V, d_G) is a metric space since d_G satisfies (i) $d_G(x, x) = 0$ for all $x \in V$, (ii) $d_G(x, y) = d_G(y, x)$ for all $x, y \in V$ and (iii) $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$ for all $x, y, z \in V$. A vertex $v \in V$ is said to *resolve* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subseteq V$ is said to be a *metric generator* for G if any pair of vertices of G is resolved by some element of S. A minimum cardinality metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

Motivated by the problem of uniquely determining the location of an intruder in a network, by means of a set of devices each of which can detect its distance to the intruder, the concepts of a metric generator and metric basis of a graph were

* Corresponding author.

E-mail addresses: yunior.ramirez@urv.cat (Y. Ramírez-Cruz), o.oellermann@uwinnipeg.ca (O.R. Oellermann), juanalberto.rodriguez@urv.cat (J.A. Rodríguez-Velázquez).

http://dx.doi.org/10.1016/j.dam.2015.06.012
0166-218X/

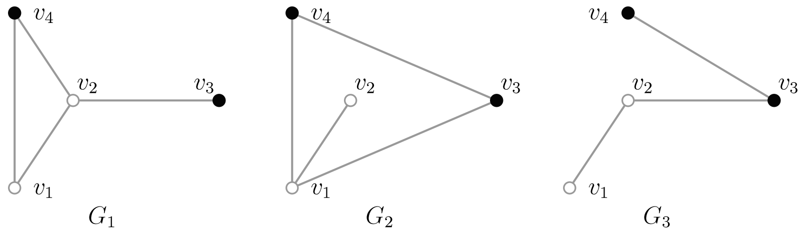

Fig. 1. The set $\{v_3, v_4\}$ is a simultaneous metric basis of $\{G_1, G_2, G_3\}$. Thus, $\mathrm{Sd}(G_1, G_2, G_3) = 2$.

introduced by Slater in [22] where metric generators were called *locating sets*. Harary and Melter independently introduced the same concept in [8], where metric generators were called *resolving sets*. Applications of the metric dimension to the navigation of robots in networks are discussed in [16] and applications to chemistry in [4,13,14]. This invariant was studied further in a number of other papers including, for instance [1,3–7,9–12,17–21,23].

The navigation problem proposed in [16] deals with the movement of a robot in a "graph space". The robot can locate itself by the presence of distinctively labeled "landmarks" in the graph space. On a graph, there is neither the concept of direction nor that of visibility. Instead, it was assumed in [16] that a robot navigating on a graph can sense the distances to a set of landmarks. If the robot knows its distances to a sufficiently large number of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph G, what are the fewest number of landmarks needed, and where should they be located, so that the distances to the landmarks uniquely determine the robot's position on G? This problem is thus equivalent to determining the metric dimension and a metric basis of G.

In this article we consider the following extension of this problem. Suppose that the topology of the navigation network may change within a range of possible graphs, say $G_1, G_2, \ldots, G_k$. This scenario may reflect the use of a dynamic network whose links change over time, etc. In this case, the above mentioned problem becomes that of determining the minimum cardinality of a set S of vertices which is simultaneously a metric generator for each graph G_i, $i \in \{1, \ldots, k\}$. So, if S is a solution to this problem, then the position of a robot can be uniquely determined by the distance to the elements of S, regardless of the graph G_i that models the network along whose edges the robot moves at each moment.

On the other hand the graphs $G_1, G_2, \ldots, G_k$ may also be the topologies of several communication networks on the same set of nodes. These communication networks may, for example, operate at different frequencies. In this case a set S of nodes that resolves each G_i would allow us to uniquely determine the location of an intruder into this family of networks.

Given a family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of (not necessarily edge-disjoint) connected graphs $G_i = (V, E_i)$ with common vertex set V (the union of whose edge sets is not necessarily the complete graph), we define a *simultaneous metric generator* for $\mathcal{G}$ to be a set $S \subseteq V$ such that S is simultaneously a metric generator for each G_i. We say that a smallest simultaneous metric generator for $\mathcal{G}$ is a *simultaneous metric basis* of $\mathcal{G}$, and its cardinality the *simultaneous metric dimension* of $\mathcal{G}$, denoted by $\mathrm{Sd}(\mathcal{G})$ or explicitly by $\mathrm{Sd}(G_1, G_2, \ldots, G_k)$. An example is shown in Fig. 1 where $\{v_3, v_4\}$ is a simultaneous metric basis of $\{G_1, G_2, G_3\}$.

The study of simultaneous parameters in graphs was introduced by Brigham and Dutton in [2], where they studied simultaneous domination. This should not be confused with studies on families sharing a constant value on a parameter, for instance the study presented in [11], where several graph families all of whose members have the same metric dimension are studied.

We will use the notation K_n, C_n, N_n and P_n to denote a complete graph, a cycle, an empty graph, and a path of order n, respectively. Let G be a graph and u, v vertices of G. We use $u \sim v$ to indicate that u is adjacent with v and $u \nsim v$ to indicate that u is not adjacent with v. The *diameter* of a graph G, denoted by $D(G)$, is the maximum distance between a pair of vertices in G. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2. General bounds

Observation 1. *For any family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of connected graphs with common vertex set V and any subfamily $\mathcal{H}$ of $\mathcal{G}$,*

$$\mathrm{Sd}(\mathcal{H}) \leq \mathrm{Sd}(\mathcal{G}) \leq min \left\{ |V| - 1, \sum_{i=1}^{k} \dim(G_i) \right\}.$$

In particular,

$$\max_{i \in \{1,\ldots,k\}} \{\dim(G_i)\} \leq \mathrm{Sd}(\mathcal{G}).$$

The above inequalities are sharp. For instance, for the family of graphs shown in Fig. 1 we have $\mathrm{Sd}(G_1, G_2, G_3) = 2 = \dim(G_1) = \dim(G_2) = \max_{i \in \{1,2,3\}}\{\dim(G_i)\}$, while for the family of graphs shown in Fig. 2 we have $\mathrm{Sd}(G_1, G_2, G_3) = 3 = |V| - 1$.

Please cite this article in press as: Y. Ramírez-Cruz, et al., The simultaneous metric dimension of graph families, Discrete Applied Mathematics (2015), http://dx.doi.org/10.1016/j.dam.2015.06.012