Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On r-hued coloring of planar graphs with girth at least 6

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ARTICLE INFO

Article history:
Received 27 January 2015
Received in revised form 5 May 2015
Accepted 10 May 2015
Available online xxxx

Keywords:
(k, r)-coloring
r-hued coloring
Girth
Planar graph

ABSTRACT

For integers $k, r > 0$, a (k, r)-coloring of a graph G is a proper k-coloring c such that for any vertex v with degree $d(v)$, v is adjacent to at least $\min\{d(v), r\}$ different colors. Such coloring is also called as an r-hued coloring. The *r-hued chromatic number* of G, $\chi_r(G)$, is the least integer k such that a (k, r)-coloring of G exists. In this paper, we proved that if G is a planar graph with girth at least 6, then $\chi_r(G) \leq r + 5$. This extends a former result in Bu and Zhu (2012). It also implies that a conjecture on r-hued coloring of planar graphs is true for planar graphs with girth at least 6.

1. Introduction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [1]. Thus $\Delta(G)$, $\delta(G)$, $g(G)$ and $\chi(G)$ denote the maximum degree, the minimum degree, the girth and the chromatic number of a graph G, respectively. When no confusion on G arises, we often use Δ for $\Delta(G)$. For $v \in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v in G, $N_G[v] = N_G(v) \cup \{v\}$, and $d_G(v) = |N_G(v)|$. When G is understood from the context, the subscript G is often omitted in these notations.

Let k, r be integers with $k > 0$ and $r > 0$, and let $[k] = \{1, 2, \ldots, k\}$. If $c : V(G) \mapsto [k]$ is a mapping, and if $V' \subseteq V(G)$, then define $c(V') = \{c(v) | v \in V'\}$. A (k, r)-*coloring* of a graph G is a mapping $c : V(G) \mapsto [k]$ satisfying both the following.

(C1) $c(u) \neq c(v)$ for every edge $uv \in E(G)$;
(C2) $|c(N_G(v))| \geq \min\{d_G(v), r\}$ for any $v \in V(G)$.

The condition (C2) is often referred to as the r-hued condition. Such coloring is also called as an r-hued coloring. For a fixed integer $r > 0$, the *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k, r)-coloring. The concept was first introduced in [10] and [6], where $\chi_2(G)$ was called *the dynamic chromatic number* of G. The study of r-hued-colorings can be traced a bit earlier, as the square coloring of a graph is the special case when $r = \Delta$.

By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and $\chi_\Delta(G) = \chi(G^2)$, where G^2 is the square graph of G. Thus r-hued coloring is a generalization of the classical vertex coloring. For any integer $i > j > 0$, any (k, i)-coloring of G is also a (k, j)-coloring of G, and so

$$\chi(G) \leq \chi_2(G) \leq \cdots \leq \chi_r(G) \leq \cdots \leq \chi_\Delta(G) = \chi_{\Delta+1}(G) = \cdots = \chi(G^2).$$

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http://dx.doi.org/10.1016/j.dam.2015.05.015

In [9], it was shown that (3, 2)-colorability remains NP-complete even when restricted to planar bipartite graphs with maximum degree at most 3 and with arbitrarily high girth. This differs considerably from the well-known result that the classical 3-colorability is polynomially solvable for graphs with maximum degree at most 3.

The r-hued chromatic numbers of some classes of graphs are known. For example, the r-hued chromatic numbers of complete graphs, cycles, trees and complete bipartite graphs have been determined in [5]. In [6], an analogue of Brooks Theorem for χ_2 was proved. It was shown in [3] that $\chi_2(G) \le 5$ holds for any planar graph G. A *Moore graph* is a regular graph with diameter d and girth $2d+1$. Ding et al. [4] proved that $\chi_r(G) \le \Delta^2 + 1$, where equality holds if and only if G is a Moore graph, which was improved to $r\Delta + 1$ in [8]. Wegner [12] conjectured that if G is a planar graph, then

$$\chi_\Delta(G) = \begin{cases} \Delta(G) + 5, & \text{if } 4 \le \Delta(G) \le 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1, & \text{if } \Delta(G) \ge 8. \end{cases}$$

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by edge contraction, and G is called H*-minor free* if G does not have H as a minor.

Define

$$K(r) = \begin{cases} r + 3, & \text{if } 2 \le r \le 3; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 4. \end{cases}$$

Lih et al. proved the following towards Wegner's conjecture.

Theorem 1.1 (*Lih et al. [7]*)**.** *Let G be a K_4-minor free graph. Then*

$$\chi_\Delta(G) \le K(\Delta(G)).$$

Song et al. extended this result by proving the following theorem. Theorem 1.1 is the special case when $r = \Delta$ of Theorem 1.2.

Theorem 1.2 (*Song et al. [11]*)**.** *Let G be a K_4-minor free graph. Then $\chi_r(G) \le K(r)$.*

A conjecture similar to the above-mentioned Wegner's conjecture is proposed in [11].

Conjecture 1.3. *Let G be a planar graph. Then*

$$\chi_r(G) \le \begin{cases} r + 3, & \text{if } 1 \le r \le 2 \\ r + 5, & \text{if } 3 \le r \le 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 8. \end{cases}$$

In this paper, we prove the following theorem.

Theorem 1.4. *If $r \ge 3$ and G is a planar graph with $g(G) \ge 6$, then $\chi_r(G) \le r + 5$.*

When $r \ge 8$, we have $r + 5 \le \lfloor 3r/2 \rfloor + 1$. Thus Theorem 1.4, together with Theorem 1.1 of [3] with $1 \le r \le 2$, justifies Conjecture 1.3 for all planar graphs with girth at least 6. Bu and Zhu in [2] proved the special case when $r = \Delta$ of Theorem 1.4, and so Theorem 1.4 is a generalization of this former result in [2].

2. Notations and terminology

Let G denote a planar graph embedded on the plane and $k > 0$ be an integer. We use $F(G)$ to denote the set of all faces of this plane graph G. For a face $f \in F(G)$, if v is a vertex on f (or if e is an edge on f, respectively), then we say that v (or e, respectively) is incident with f. The number of edges incident with f is denoted by $d_G(f)$, where each cut edge counts twice. A face f of G is called a *k-face* (or a *k^+-face*, respectively) if $d_G(f) = k$ (or $d_G(f) \ge k$, respectively). A vertex of degree k (at least k, at most k, respectively) in G is called a *k-vertex* (*k^+-vertex, k^--vertex*, respectively). We use $n_i(v)$ to denote the number of i-vertices adjacent to v.

For two vertices $u, w \in V(G)$, we say that u and w are *weak-adjacent* if there is a 2-vertex v such that $u, w \in N_G(v)$. A 3-vertex v is a *weak 3-vertex* if v is adjacent to a 2-vertex. The neighbors of a weak 3-vertex are called *star-adjacent*. If a 5-vertex is weak-adjacent to five 5-vertices, we call it a *bad vertex*. (As an example, see the vertex v in H_4 of Fig. 2). If a 5-vertex is adjacent to one weak 3-vertex and is weak-adjacent to four other 5-vertices, we call it a *semi-bad* type vertex. As Fig. 2 demonstrates, the vertex v in H_5 is a semi-bad type vertex.

Let G be a graph with $V = V(G)$, and let $V' \subseteq V$ be a vertex subset. As in [1], $G[V']$ is the subgraph of G induced by V'. A mapping $c : V' \to [k]$ is a *partial* (k, r)*-coloring* of G if c is a (k, r)-coloring of $G[V']$. The subset V' is the *support* of the partial (k, r)-coloring c. The support of c is denoted by $S(c)$. If c_1, c_2 are two partial (k, r)-colorings of G such that $S(c_1) \subseteq S(c_2)$ and such that for any $v \in S(c_1)$, $c_1(v) = c_2(v)$, then we say that c_2 is an *extension* of c_1. Given a partial (k, r)-coloring c on $V' \subset V(G)$, for each $v \in V - V'$, define $\{c(v)\} = \emptyset$; and for every vertex $v \in V$, we extend the definition of $c(N_G(v))$ by setting $c(N_G(v)) = \cup_{z \in N_G(v)} \{c(z)\}$, and define

$$c[v] = \begin{cases} \{c(v)\}, & \text{if } |c(N_G(v))| \ge r; \\ \{c(v)\} \cup c(N_G(v)), & \text{otherwise.} \end{cases} \quad (1)$$

Please cite this article in press as: H. Song, et al., On r-hued coloring of planar graphs with girth at least 6, Discrete Applied Mathematics (2015), http://dx.doi.org/10.1016/j.dam.2015.05.015