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The channel capacity of read/write isolated memory[☆]

Chuanlong Wang[a], Xuerong Yong[b,*], Mordecai Golin[c]

[a] Department of Mathematics, Taiyuan Normal University, Taiyuan, Shanxi, China

[b] Department of Mathematical Sciences, The University of Puerto Rico, Mayaguez, PR 00681, USA

[c] Department of Computer Science and Engineering, Hong Kong University of Science & Technology, Clear Water Bay, Hong Kong

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ABSTRACT

A read/write isolated memory is a binary re-writable medium in which (i) two consecutive locations cannot both store 1's and also in which (ii) two consecutive locations cannot both be modified during the same rewriting pass. Its channel capacity C, in bits per symbol per rewrite, is defined as

$$C = \lim_{k,r\to\infty} \frac{\log_2 N(k,r)}{kr},$$

where k is the size of the memory in binary symbols, r is the lifetime of the memory in rewriting cycles, and $N(k,r)$ is the number of distinct sequences of r-characters that satisfy the constraints. This quantity was originally considered by Cohn (1995) who proved that $0.509\ldots \le C \le 0.560297\ldots$ and conjectured that $C = 0.537\ldots$. Subsequently, Golin et al. (2004) refined the bounds to $0.53500\ldots \le C \le 0.55209\ldots$ and conjectured that $C = 0.5350\ldots$.

In this paper, we develop a new technique for computing C as a particular type of constrained binary matrix and obtain that

$$C = 0.53501\ldots.$$

The methods introduced in this note are not specific to this particular problem but can also be used to consider various other computational counting problems.

1. Introduction

Binary (0, 1) sequences in a communication channel are often required to satisfy predefined specific constraints that guarantee reliable storage or transmission. The set of all permissible binary memory configurations of a given size can be viewed as a channel alphabet that obeys specific restrictions. A *read/write isolated memory* (RWIM) is a binary, linearly ordered, re-writable storage medium satisfying two restrictions. The first, the *read restriction* states that no two consecutive positions in the memory may both store 1's. The second, the *write restriction*, states that when the memory is rewritten no two consecutive positions in the memory are allowed to change. A fixed size memory can be viewed as a character sent over

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* Corresponding author.
E-mail addresses: clwang218@126.net (C. Wang), xryong@dimacs.rutgers.edu (X. Yong), golin@cse.ust.hk (M. Golin).

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$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Fig. 1. Matrix B_1 satisfies the RWIM constraints. Matrix B_2 does not for two reasons: the two consecutive bold 1's in the first row contradict the read restriction and the bold 2×2 submatrix in the fourth and fifth columns contradicts the write restriction.

a noiseless communication channel; the memory contents after rewriting pass i is the ith character sent over the channel. The write restrictions dictate which characters may follow which characters in the channel.

The *channel capacity* for a rewritable memory, measured in bits per character was studied in [20,21]. The capacity of memory, in bits per rewrite, is defined as

$$C_k = \lim_{r\to\infty} \frac{1}{r} \log_2 N(k, r),$$

where k is the size of the memory in binary symbols, r is the lifetime of the memory in rewriting cycles, and $N(k, r)$ is the number of distinct sequences of r-characters that satisfy the constraints. Note that $N(k, r)$ is equal to the number of distinct paths through a channel graph that describes permissible transitions among characters. Shannon showed that $C_k = \log_2 \lambda_k$, where λ_k is the spectral radius of the channel graph, *i.e.*, the largest eigenvalue of its adjacency matrix. (From the Perron–Frobenius Theorem in the theory of nonnegative matrices [1], λ_k is a positive number.) Shannon also proved that the capacity is an upper bound on the rate achievable by any coding scheme. A code is nearly *optimal* if the capacity obtained from computation is very close to its true value.

The channel capacity C of RWIM, in bits per symbol per rewrite, can be defined as [7,11]

$$C = \lim_{k,r\to\infty} \frac{\log_2 N(k, r)}{kr}. \tag{1}$$

The limit has been proven to exist but the exact value of C is difficult to obtain.

In applications the codes with the read and/or write restrictions above are typical of those that used magnetic recording and optical recording. To the best of our knowledge, Freiman and Wyner [8] were the first to consider read isolated memories, followed by Kautz [14]. Write isolated memories were originally examined by Robinson [18] and then Cohen [6]. This memory is used in the context of an asymmetric error-correcting ternary code and re-writable optical disc. They showed independently that the two different types of memories have the same capacity $\log_2 \varphi = 0.694\ldots$, in bits per symbol, in which φ is the larger root of the *Fibonacci recurrence*: $F_{n+2} = F_{n+1} + F_n$.

A (d, k)-*Runlength Limited (RLL)* code over the binary alphabet $\{0, 1\}$ has d and k being the minimum and maximum permitted numbers of 0's separating consecutive 1's in its each codeword, respectively. This class of codes has wide applications [13]. As examples, the $(1, 3)$-RLL constraint is often used in magnetic disc drivers and the $(2, 10)$-RLL is in compact audio discs. The $(1, \infty)$-RLL is the read isolated constraint.

The (d, k)-RLL codes were sets of one dimensional strings. A 2-*D* code is a set of 2-*D* arrays/matrices that must each satisfy a set of constraints. The constraints are often given as horizontal (on the rows) and vertical (on the columns). Let $N(k, r)$ now be the number of $n \times r$ matrices that satisfy the given constraints. The capacity per bit of the 2-*D* code is given by

$$C = \lim_{k,r\to\infty} \frac{\log_2 N(k, r)}{kr}. \tag{2}$$

A 2-*D* (d, k)-RLL code is a matrix whose rows and columns all individually satisfy the 1-*D*(d, k)-RLL constraint. Over the last two decades the capacity of these codes have started to be studied. A relatively recent contribution is [9].

Notice that (2) is exactly in the same form as (1) so channel capacity looks very similar to 2-*D* code capacity. In fact, the RWIM channel can actually be rewritten as a 2-*D* code as follows. For a memory of size n with lifetime r create an $r \times k$ binary matrix B in which row i is the contents of the memory at time i. B must satisfy the following two constraints (see Fig. 1 for examples):

1. *Read restriction:* B does not contain any two horizontally consecutive ones, i.e., it does not contain any 1×2 submatrix $(1, 1)$.
2. *Write restriction:* B does not contain any 2×2 submatrix of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Any B that satisfies these two constraints represents a legal set of rewrites for a r bit memory in the RWIM channel so $N(k, r)$ for the RWIM also represents the number of 2-*D* codewords that satisfy the read and write restrictions above and the RWIM channel capacity is the same as the capacity of the 2-*D* RWIM codes.

When Cohen originally investigated the capacity of the RWIM constraint in [7] he derived the following upper and lower bounds on the capacity C:

$$0.509\ldots \le C \le 0.560297\ldots.$$