Discrete Applied Mathematics 198 (2016) 291–296

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

A note on equidistant subspace codes

Daniele Bartoli [a], Francesco Pavese [b,*]

[a] Department of Mathematics, Ghent University, Krijgslaan 281, 9000 Ghent, Belgium

[b] Dipartimento di Matematica, Informatica ed Economia, University of Basilicata, Contrada Macchia Romana, 85100 Potenza, Italy

ARTICLE INFO

Article history:
Received 26 January 2015
Received in revised form 18 May 2015
Accepted 21 June 2015
Available online 9 July 2015

Keywords:
Equidistant subspace code
Hyperbolic quadric

ABSTRACT

Equidistant subspace codes are studied. A classification of the largest 1-intersecting codes in PG(5, 2), whose codewords are planes, is provided. Also, new constructions of large equidistant codes are presented.

1. Introduction

Let V be an r-dimensional vector space over GF(q), q any prime power. The set $S(V)$ of all subspaces of V, or subspaces of the projective space PG(V), forms a metric space with respect to the *subspace distance* defined by $d_s(U, U') = \dim(U + U') - \dim(U \cap U')$. In the context of subspace codes, the main problem is to determine the largest possible size of codes in the space $(S(V), d_s)$ with a given minimum distance, and to classify the corresponding optimal codes. The interest in these codes is a consequence of the fact that codes in the projective space and codes in the Grassmannian over a finite field referred to as subspace codes and constant-dimension codes, respectively, have been proposed for error control in random linear network coding, see [24]. For general results on bounds and constructions of constant-dimension subspaces codes, see [5,6,13–17,22,23,28,30].

In this note we are interested in equidistant constant-dimension subspace codes. An *equidistant constant-dimension subspace code* or *t-intersecting code*, is a collection $\mathcal{C}$ of $(k-1)$-dimensional projective subspaces of PG$(r-1, q)$ mutually intersecting in a $(t-1)$-dimensional projective space, where $t < k$. The largest equidistant constant-dimension subspace code is said to be *optimal*. In this context interesting constructions of such codes were given in [12,18,19]. An important concept in this context is the sunflower. A *sunflower* $\mathcal{S}$ is a t-intersecting code in which any two elements of $\mathcal{S}$ intersect in the same $(t-1)$-dimensional projective space. For a code $\mathcal{C}$ we define $\mathcal{C}^\perp$ as the code which consists of the dual subspaces of $\mathcal{C}$. Then it is easily seen that $\mathcal{C}$ is a t-intersecting code, whose codewords are $(k-1)$-dimensional projective subspaces if and only if $\mathcal{C}^\perp = \{X^\perp \mid X \in \mathcal{C}\}$ is an $(r-2k+t)$-intersecting code, whose codewords are $(r-k-1)$-dimensional projective subspaces.

As noted in [12], a known upper bound in coding theory [9,10], can be adapted for equidistant constant-dimension subspace codes obtaining that if a t-intersecting code, whose codewords are $(k-1)$-dimensional projective subspaces, has more than $\left(\frac{q^k-q^t}{q-1}\right)^2 + \frac{q^k-q^t}{q-1} + 1$ codewords, then the code is a sunflower. Therefore, if r is large enough, an optimal equidistant constant-dimension subspace code is a sunflower.

* Corresponding author.
E-mail addresses: dbartoli@cage.ugent.be (D. Bartoli), francesco.pavese@unibas.it (F. Pavese).

http://dx.doi.org/10.1016/j.dam.2015.06.017

On the other hand a conjecture, attributed to Deza, states that if a t-intersecting code, whose codewords are $(k-1)$-dimensional projective subspaces, has more than $|\mathrm{PG}(k,q)|$ codewords, then the code is a sunflower. In [12], the authors exhibit a set of sixteen planes in PG(5, 2) mutually intersecting in a point that therefore is a counterexample to Deza's Conjecture. In Section 2 we prove that the maximum number of planes in PG(5, 2) mutually intersecting in a point is twenty. We also provide an example and prove that such a family is unique up to collineations. Notice that in [1] a classification of optimal 1-intersecting codes in PG(5, q), $q > 2$, whose codewords are planes, was obtained.

In Section 3 constructions of large equidistant constant-dimension subspace codes from subspaces of constant rank matrices are discussed.

2. Optimal non-sunflower equidistant codes

In PG(5, q) a *Klein quadric* is the set of singular points for some non-degenerate quadratic form of hyperbolic type defined on the underlying vector space. The projective spaces of maximal dimension contained in a Klein quadric are planes and are also called *generators*. The set of all generators of a Klein quadric is divided into two distinct subsets of the same size, called *systems of generators*. Two distinct generators from the same system meet in a point, two generators from different systems meet either in a line or they are disjoint. See [20] for more details.

In [1] the authors considered set of planes of a projective space with the property that any two of them intersect in exactly a point. Their main result is the following:

Theorem 2.1. *Let $\mathcal{C}$ be a set of planes in* PG(5, q) *mutually intersecting in a point, spanning* PG(5, q) *and that is not a sunflower. If $q \geq 3$ and $|\mathcal{C}| \geq 3(q^2+q+1)$, then $\mathcal{C}$ is a subset of the $(q+1)(q^2+1)$ planes forming a system of generators of a Klein quadric.*

In other words they proved that, if $q > 2$, an optimal 1-intersecting code in PG(5, q), whose codewords are planes, is a system of generators of a Klein quadric. Moreover, they also proved that no larger 1-intersecting code in PG(d, q), whose codewords are planes, spanning PG(d, q) and that is not a sunflower, exists whenever $d \geq 6$. In the remaining part of this section we deal with the case $q = 2$.

Theorem 2.2. *Let $\mathcal{C}$ be a set of s planes in* PG(r, 2)*, with $s > 19$, which is not a sunflower. Then $\mathcal{C}$ is contained in* PG(5, 2).

Proof. Let P be a point belonging to a plane of $\mathcal{C}$. Since we are supposing that $\mathcal{C}$ is not a sunflower, then there exists a plane π of $\mathcal{C}$ not passing through P. This implies that the maximum number of planes through P is 7, since all these planes must intersect π in distinct points. If there was no point contained in more than three planes, then the total number of planes is at most $7 \times 2 + 1 = 15$. Therefore there exists at least a point P_0 contained in at least four planes. In the following we will denote with $\overline{\tau}$ or τ_i any plane through P_0 and with σ_j or σ any plane not containing P_0. Note that any two planes τ_i span a 4-space, otherwise they would have a line in common. We will denote the points P_i as the points having, in homogeneous coordinates, all zeros and 1 in position i. We distinguish several cases.

(1) $\dim(\langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle) = 4$. First of all notice that in this case all the planes σ_j must be contained in $\Pi_4 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$. Suppose now that there exists a plane $\tau_{\bar{i}} \not\subset \Pi_4$. Since it must meet all the planes σ_j, it meets Π_4 in a line $\tau_{\bar{i}} \cap \Pi_4 = \ell = \{P_0, Q, R\}$. We already know that at most six planes σ_j pass through Q and at most six planes σ_j pass through R, other than $\tau_{\bar{i}}$. Moreover, any plane σ_j must intersect $\tau_{\bar{i}}$ either in Q or in R, since it is contained in Π_4. The total number of planes σ_j is then 12; the number of planes τ_i is 7 and therefore the maximum number of planes of $\mathcal{C}$ in this case is $12 + 7 = 19$. Since we are supposing that the number of planes of $\mathcal{C}$ is at least 20, this is a contradiction. Then all the planes τ_i are subsets of Π_4.
We now claim that there exists no $\tau \notin \{\tau_1, \tau_2, \tau_3, \tau_4\}$ passing through P_0. Assume for contradiction that there exists a plane τ such that $\tau \notin \{\tau_1, \tau_2, \tau_3, \tau_4\}$ which passes through P_0. Then $\tau, \tau_1, \tau_2, \tau_3, \tau_4$ partition $\Pi_4 \setminus \{P_0\}$. A plane σ not through P_0 intersects at least one among $\tau, \tau_1, \tau_2, \tau_3, \tau_4$ in a line. This is impossible. Then the unique planes through P_0 are $\tau_1, \tau_2, \tau_3, \tau_4$. The same argument holds for each point of Π_4. Therefore through each point there pass no more than 4 planes. This means that the maximum number of planes of this configuration is
$$\frac{31 \times 4}{7} < 18.$$
(2) $\dim(\langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle) = 5$. Let $\Pi_5 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$. Arguing as before, if there was $\tau_{\bar{i}} \not\subset \Pi_5$, then $\tau_{\bar{i}} \cap \Pi_5 = \ell = \{P_0, Q, R\}$ and at most six planes σ_j pass through Q and at most six planes σ_j pass through R, other than $\tau_{\bar{i}}$. Since any plane σ_j must be contained in Π_5, then it must intersect $\tau_{\bar{i}}$ either in Q or in R. Therefore the maximum number of planes in this case is $12 + 7 = 19$. Since by hypothesis the number of the planes is greater than 19, then all the planes through P_0 must be contained in Π_5. Also, any plane σ_j is contained in Π_5.
(3) $\dim(\langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle) = 6$.
    (a) Suppose that $\forall i_1, i_2, i_3 \in \{1, 2, 3, 4\}$ $\dim(\langle \tau_{i_1}, \tau_{i_2}, \tau_{i_3} \rangle) = 6$. Consider a plane $\sigma_j = \langle Q_1, Q_2, Q_3 \rangle$, where $Q_k \in \tau_{i_k}$, $k = 1, 2, 3$. Clearly, Q_1, Q_2, Q_3 are in general position, since $\tau_{i_1}, \tau_{i_2}, \tau_{i_3}$ generate a space of dimension 6. Let $\overline{\tau} \in \{\tau_1, \tau_2, \tau_3, \tau_4\} \setminus \{\tau_{i_1}, \tau_{i_2}, \tau_{i_3}\}$. By assumption, every three planes among $\tau_1, \tau_2, \tau_3, \tau_4$ generate a space of dimension 6,