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On a conjecture on the order of cages with a given girth pair

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ABSTRACT

A $(k; g, h)$-graph is a k-regular graph of girth pair (g, h) where g is the girth of the graph, h is the length of a smallest cycle of different parity than g and $g < h$. A $(k; g, h)$-cage is a $(k; g, h)$-graph with the least possible number of vertices denoted $n(k; g, h)$. Harary and Kóvacs (1983) conjectured the inequality $n(k; g, h) \leq n(k, h)$ for all $k \geq 3$, $g \geq 3$, $h \geq g + 1$. In this paper, we prove this conjecture for all $(k; g, h)$-cage with g odd provided that a bipartite (k, h)-cage exists. When g is even we prove the conjecture for $h \geq 2g - 1$, provided that a bipartite (k, g)-cage exists.

1. Introduction

In [11], Harary and Kóvacs generalize the concept of (k, g)-cages by replacing the girth condition with a *girth pair condition* (g, h) (i.e. g is the girth of the graph, h is the length of a smallest cycle of different parity than g and $g < h$). In that work the authors proved the existence of $(k; g, h)$-cages with $3 \leq g < h$, obtaining the following inequality: $n(k; g, h) \leq 2n(k, h)$. Also, they proved that if $k \geq 3$ and $h \geq 4$, then $n(k; h - 1, h) \leq n(k, h)$, and established the following conjecture.

Conjecture 1.1 ([11]). *$n(k; g, h) \leq n(k, h)$ for all $k \geq 3$, $g \geq 3$, $h \geq g + 1$.*

The exact values $n(k; 4, h)$ are studied in [14,16,20] and exact values of $n(3; 6, h)$ for $h = 7, 9, 11$ are determined in [5]. All these values support Conjecture 1.1. In [19] it is proved the strict inequality $n(k; h - 1, h) < n(k, h)$ for $k \geq 3$ and $h \geq 4$.

We want to emphasize that every known (k, g)-cage with even girth g is bipartite and it is conjectured that all cages with even girth are bipartite [15,18]. In this regard, there is a result (cf. [4]), that states that all (k, g)-cages with girth g even and such that have *excess* $e = n(k, g) - n_0(k, g) \leq k - 2$ are bipartite. Hence, the requirement of the existence of a bipartite (k, g)-cage for even g is natural.

In the first part of the paper, we settle Conjecture 1.1 when the smallest girth g is odd provided that there is a bipartite (k, h)-cage with $g < h$. We also prove the exact value $n(3; 5, 8) = 18$.

In the second part, we study Conjecture 1.1 when the smallest girth g is even, and we prove the strict inequality $n(k; g, h) < n(k, h)$ if $h \geq 2g - 1$ provided that there is a bipartite (k, g)-cage. As a consequence, we prove the inequality for girth $g = 6, 8, 12$ and $k = q + 1$, where q is a prime power and also for (k, g)-cages with small excess since all these graphs are bipartite [8].

2. Terminology and known results

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [6].

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Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $U \subset V$ the subgraph induced by U is denoted as $G[U]$. The distance $d_G(u, v)$ between two vertices u and v is the minimum length of a path from u to v (*uv-path*) in G. The *girth* of a graph G is the length $g = g(G)$ of a shortest cycle. A *girdle* is a shortest cycle. The *neighborhood* $N(u)$ of a vertex u is the set of its neighbors, i.e., the vertices adjacent to u. The *degree* of a vertex $u \in V$ is the cardinality of $N(u)$. A graph is called *k-regular* if all its vertices have the same degree k. A (k, g)-*graph* is a k-regular graph of girth g and a (k, g)-*cage* is a (k, g)-graph with the smallest possible number of vertices. For $k \geq 3$ and $g \geq 5$ the order $n(k, g)$ of a cage is bounded by

$$n(k,g) \geq n_0(k,g) = \begin{cases} 1 + k \displaystyle\sum_{i=0}^{(g-3)/2} (k-1)^i & g \text{ odd;} \\ 2 \displaystyle\sum_{i=0}^{(g-2)/2} (k-1)^i & g \text{ even.} \end{cases} \quad (1)$$

This bound is known as the *Moore bound* for cages. Note that cages of even girth contain a tree of depth $(g-2)/2$ and order $n_0(k, g) = 2 \sum_{i=0}^{(g-2)/2} (k-1)^i$ rooted in any edge uv.

A key point for proving many results in cages is the so-called *Girth Monotonicity Theorem*, established by Erdős and Sachs [7] and Fu et al. [9].

Theorem 2.1 ([7,9]). *Let $k \geq 2$, $3 \leq g_1 < g_2$ be integers. Then $n(k, g_1) < n(k, g_2)$.*

The following useful lemma is a consequence of Theorem 2.1.

Lemma 2.1 ([13]). *Let G be a (k, g)-cage with $k \geq 3$ and girth $g \geq 4$. Then every edge of G lies on at least $k-1$ cycles of length at most $g+1$.*

3. Results

In what follows we need the following notation. For $uv \in E(G)$ and $l \geq 0$, let denote the sets

$$B^l_{uv} = \{x \in V(G) : d(x, u) = l \text{ and } d(x, v) = l + 1\} \quad \text{and} \quad \overline{B}^l_{uv} = \bigcup_{i=0}^{l} B^i_{uv}.$$

Observe that $B^0_{uv} = \{u\} = \overline{B}^0_{uv}$ and $B^1_{uv} = N(u) - v$ while $\overline{B}^1_{uv} = (N(u) - v) \cup \{u\}$. Moreover, note that $B^l_{uv} \neq B^l_{vu}$ and $\overline{B}^l_{uv} \neq \overline{B}^l_{vu}$.

Let denote $T^l_{uv} = G[\overline{B}^l_{uv} \cup \overline{B}^l_{vu}]$ and observe that if $l \leq g/2 - 2$, where g is the girth of G, then T^l_{uv} is the tree of depth l rooted in the edge uv. When $l = g/2 - 1$ the subgraph T^l_{uv} may not be a tree, it can contain edges between vertices in B^l_{uv} and vertices in B^l_{vu}.

3.1. Conjecture 1.1 holds for girth pair (g, h) with g odd

Lemma 3.1. *Let G be a bipartite (k, h)-cage with $k \geq 3$ and even girth $h > 6$. Then there exist an edge $uv \in E(G)$ and a girdle β in G such that $V(\beta) \cap \overline{B}^{\lfloor h/4 \rfloor - 2}_{uv} = \emptyset$ and $V(\beta) \cap \overline{B}^{\lfloor h/4 \rfloor - 1}_{vu} = \emptyset$.*

Proof. Let $\alpha = w_0 w_1 \cdots w_\ell z_\ell z_{\ell-1} \cdots z_0 w_0$ be a girdle of G and take the subgraph $T^\ell_{w_0 z_0}$ for $\ell = h/2 - 1$. Then $z_t \in V(\alpha) \cap B^t_{z_0 w_0}$ and $w_t \in V(\alpha) \cap B^t_{w_0 z_0}$ for $t = 0, 1, \ldots, \ell = h/2 - 1$. From Lemma 2.1, it follows that there is another girdle $\beta \neq \alpha$ of G such that $w_\ell z_\ell \in E(\beta)$.

Let $r = \min\{i : V(\beta) \cap B^i_{w_0 z_0} \neq \emptyset\}$ and $t = \min\{i : V(\beta) \cap B^i_{z_0 w_0} \neq \emptyset\}$, $x \in V(\beta) \cap B^r_{w_0 z_0}$ and $y \in V(\beta) \cap B^t_{z_0 w_0}$. Observe that the lemma holds if $r \geq \lfloor h/4 \rfloor - 1$ and $t \geq \lfloor h/4 \rfloor$ for the edge uv taking $u = w_0$ and $v = z_0$.

Suppose $r \leq \lfloor h/4 \rfloor - 2$. Since α is the unique girdle containing both edges $w_0 z_0$ and $w_\ell z_\ell$, the girdle β through the edge $w_\ell z_\ell$ must also contain a path $P_{w_\ell x}$, a path P_{xa} where $a \in B^\ell_{w_0 z_0} - w_\ell$, a path $P_{y z_\ell}$, and a path P_{by} where $b \in B^\ell_{z_0 w_0} - z_\ell$. Moreover, since $a \neq b$ the girdle β must also contain a path P_{ab} disjoint from the above ones. Since $|E(P_{w_\ell x})|$ and $|E(P_{xa})| \geq h/2 - 1 - r$, $|E(P_{by})|$ and $|E(P_{y z_\ell})| \geq h/2 - 1 - t$, we have

$$h = |E(\beta)| \geq |w_\ell z_\ell| + 2(h/2 - 1 - r) + 2(h/2 - 1 - t) + |E(P_{ab})| \geq 2(h - t - r - 1)$$

yielding that

$$h/2 \leq t + r + 1. \quad (2)$$

Take $u = z_{\lfloor h/4 \rfloor - r - 2}$ and $v = z_{\lfloor h/4 \rfloor - r - 1}$ (see Fig. 1). Hence $x \in B^{\lfloor h/4 \rfloor - 1}_{uv}$, that is $x \notin \overline{B}^{\lfloor h/4 \rfloor - 2}_{uv}$, and by (2) we have $t \geq h/2 - (\lfloor h/4 \rfloor - 2) - 1 = \lceil h/4 \rceil + 1 > \lfloor h/4 \rfloor - 1$ implying that $y \notin \overline{B}^{\lfloor h/4 \rfloor - 1}_{vu}$. Therefore, the result holds for the edge uv and the girdle β in the case $r \leq \lfloor h/4 \rfloor - 2$. Similarly, we proceed for $t \leq \lfloor h/4 \rfloor - 1$ and the result follows. ■

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