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The b-chromatic index of direct product of graphs[☆]

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ABSTRACT

The b-chromatic index $\varphi'(G)$ of a graph G is the largest integer k such that G admits a proper k-edge coloring in which every color class contains at least one edge incident to edges in every other color class. We give in this work bounds for the b-chromatic index of the direct product of graphs and provide general results for many direct products of regular graphs. In addition, we introduce an integer linear programming model for the b-edge coloring problem, which we use for computing exact results for the direct product of some special graph classes.

1. Introduction

A b-*vertex coloring* of a graph G is a proper vertex coloring of G such that each color class contains a vertex that has at least one neighbor in every other color class. The b-*chromatic number* $\varphi(G)$ of a graph G is the largest integer k for which G has a b-vertex coloring with k colors. This concept was introduced in [17] by Irving and Manlove by considering proper colorings that are minimal with respect to a partial order P defined on the set of all partitions of the vertex set of G that induces a proper coloring. The chromatic number $\chi(G)$ is the minimum number of colors used among all minimal elements of P, while $\varphi(G)$ is the maximum number of colors among this same set.

Since then the b-chromatic number has been an active field of research. Already Irving and Manlove [17] have shown that computing $\varphi(G)$ is an NP-hard problem in general and polynomial for trees. It is known that the problem remains NP-hard even when restricted to bipartite graphs, (see [26]), and that it is also hard to approximate in polynomial time within a factor of $\frac{120}{113} - \epsilon$, for any $\epsilon > 0$ unless $P = NP$, see [9]. Analysis of the b-chromatic number for special graph classes such as powers of paths, cycles, complete binary trees, and complete caterpillars can be found in [10–12]. Further graph classes, such as cacti [7], Kneser graphs [21], cographs, P_4 sparse [4], P_4 tidy graphs [30] and large girth graphs [28,6], were more recently considered too. Bounds for the b-chromatic number were studied in turn in [2,1,8,23,25]. The b-chromatic number under graph operations was considered in [24] for the Cartesian product and in [19] for other three standard products.

The behavior of the b-chromatic number differs notoriously from the classical chromatic number. The values of k for which a graph admits a b-coloring do not necessarily form an interval in the set of integers. It makes therefore sense to study the graph families in which there exists a t–b-vertex coloring for every integer t between $\chi(G)$ and $\varphi(G)$. This interesting concept of b-*continuous graphs* was introduced by Barth et al. in [3]. Another atypical property of the b-chromatic number is that it can increase when taking induced subgraphs. A graph G is b-*monotonous* if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced

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subgraph H_1 of G and every induced subgraph H_2 of H_1. Cographs for instance are b-monotonous. This idea was proposed in [4]. In [16], b-*perfect* graphs were introduced and studied. A graph G is b-perfect if $\chi(H) = \varphi(H)$ for every induced subgraph H of G.

Intuitively, for a b-coloring to be possible on a graph G, we need to have enough vertices of high enough degree, at least one in each color class. Let $v_1, \dots, v_n$ be a sequence of vertices, such that $d(v_1) \geq \cdots \geq d(v_n)$ where $d(v_i)$ denotes the degree of v_i. Then $m(G) = \max\{i : d(v_i) \geq i - 1\}$ is an upper bound for $\varphi(G)$. From this point of view, d-regular graphs are of special interest, since $m(G) = d + 1$ for a d-regular graph G and every vertex is a candidate to have each color class in its neighborhood. Indeed, in [26] it was shown that if a d-regular graph G has at least d^4 vertices, then the equality $\varphi(G) = d+1$ holds. This bound was later improved to $2d^3$ in [5]. In particular, it was shown in [18] that there are only four exceptions among cubic graphs with $\varphi(G) < 4$, one of them being the Petersen graph.

We continue in this work the study of the edge version of the b-vertex coloring and the b-chromatic number introduced in [20], namely the b-edge coloring and the b-chromatic index, respectively. A b-*edge coloring* of a graph G is a proper edge coloring of G such that each color class contains an edge that has at least one incident edge in every other color class. The b-*chromatic index of a graph G* is the largest integer $\varphi'(G)$ for which G has a b-edge coloring with $\varphi'(G)$ colors. We say that this coloring *realizes* $\varphi'(G)$. An edge e of color i that has all other colors on its incident edges is called *color i dominating edge*; we say also that color i is *realized* on e. The trivial upper bound $m'(G)$ for the b-edge coloring is defined similarly as for the vertex version: $m'(G) = max\{i : d(e_i) \geq i - 1\}$, where $d(e_1) \geq \cdots \geq d(e_m)$ is the degree sequence of the edges $e_1, \dots, e_m$ of G.

In [20], the authors determine the b-chromatic index of trees, and give conditions for graphs that have the b-chromatic index strictly less than $m'(G)$, as well as conditions on graph G for which $\varphi'(G) = m'(G)$. They prove further that $\varphi'(G) = 5$ for connected cubic graphs, with only four exceptions: $K_4, K_{3,3}$, the prism over K_3, and the cube Q_3. Regarding the complexity of the problem, we note that determining whether $\varphi'(G) = m'(G)$ was shown to be NP-complete by Lima et al. in [27].

In the next section we recall some standard definitions and notation. In the third section we describe bounds for $\varphi'(G \times H)$ for some graph classes. In the fourth section we show some important consequences of a previous theorem of [20] that allows exact results for φ' under certain conditions.

Determining the b-chromatic index of a graph can be very tedious already for small examples. For this reason we developed a linear programming model for the problem, which we describe and study for effectiveness in the fifth section. With this method and all previous results we were able to produce exact values for some families of direct products, documented in the last section.

2. Preliminaries

Let G be a graph. The *line graph* $\mathcal{L}(G)$ of G is the graph with $V(\mathcal{L}(G)) = E(G)$, and two edges of G are adjacent in $\mathcal{L}(G)$ if they share a common vertex. Clearly $\varphi'(G) = \varphi(\mathcal{L}(G))$. The number of vertices incident with the vertex v is the *degree* of v and is denoted by $d(v)$. If all vertices have the same degree d, we say that G is a d-*regular* graph. The *degree of an edge* $e = uv$ is denoted by $d(e)$ and equals to $d(u) + d(v) - 2$. Notice that $d(e)$ in graph G equals to $d(v)$ in $\mathcal{L}(G)$ where $v \in V(\mathcal{L}(G))$ corresponds to $e \in E(G)$. We denote the set of edges incident to an edge e by $N(e)$. A graph is called an r-edge regular graph if all its edges have the same degree r.

The *direct product* $G \times H$ of graphs G and H has vertex set $V(G) \times V(H)$; two vertices (g, h) and (g', h') are adjacent in $G \times H$ if they are adjacent in both coordinates, i.e. $gg' \in E(G)$ and $hh' \in E(H)$. If $e = (g, h)(g', h') \in E(G \times H)$, let $p_G(e) = gg'$ and $p_H(e) = hh'$ be the *projection* of edge e over G and H, respectively. The direct product is associative (see [15]) and hence we can write more factors without brackets: $G_1 \times \cdots \times G_k = \times_{i=1}^{k} G_i$. The direct product seems to be the most elusive product among all four standard products (Cartesian, strong, direct and lexicographic). The reason for this is the fact that each edge of $G \times H$ projects to an edge in both factors, which is not the case on other products. This also constitutes the direct product as the product in categorical sense. Even basic graph properties, such as connectedness are non trivial for the direct product. Indeed, $G \times H$ need not be connected, even if both factors are. This happens exactly when both factors are bipartite (and connected) and in this case there are exactly two components (see [31] or [15]). Also the following distance formula (see [22]),

$$d_{G\times H}((g, h), (g', h')) = \min\{\max\{d_G^e(g, g'), d_H^e(h, h')\}, \max\{d_G^o(g, g'), d_H^o(h, h')\}\}$$

is far more complicated for the direct product than for others. Here $d_G^e(g, g')$ means the length of a shortest walk of even length between g and g' in G and $d_G^o(g, g')$ the length of a shortest odd walk between g and g' in G. If such a walk does not exist, we set $d_G^e(g, g')$ or $d_G^o(g, g')$ to be infinite. For more about direct product graphs see the book [15].

3. Bounds for $\varphi'(G \times H)$

A *one-factor* or a *perfect matching* of a graph G is a set of independent edges of G that meet every vertex of G. Clearly, a graph with a one-factor has an even number of vertices. A *one-factorization* of G is a partition of $E(G)$ into one-factors. Thus, in a one-factorization of G, every edge belongs to exactly one one-factor. Evidently G must be regular with an even number of vertices if it has a one-factorization. The basic examples of graphs with one factorization are even cycles and hypercubes. Graph products form a rich field for one factorizations, see for instance Section 30.1 of [15].