Discrete Applied Mathematics 178 (2014) 19–26

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

# A decision algorithm for reversible pairs of polygons

Jin Akiyama [a,*], David Rappaport [b], Hyunwoo Seong [c]

[a] Research Center for Math Education, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
[b] School of Computing, Queen's University, Kingston, ON K7L 3N6, Canada
[c] Department of Mathematics, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

ARTICLE INFO

*Article history:*
Received 7 March 2014
Received in revised form 17 June 2014
Accepted 19 June 2014
Available online 10 July 2014

*Keywords:*
Reversibility
Decision algorithm
Equi-decomposability
Tiling
Reversion trunk
Parallelogram

ABSTRACT

A given pair of convex polygons $\alpha$ and $\beta$ is said to be reversible if $\alpha$ has a dissection into a finite number of pieces which can be rearranged to form $\beta$ under some conditions. In this paper, we give an algorithm to decide whether or not a given pair of polygons $\alpha$ and $\beta$ is reversible. Furthermore, a method of how to dissect $\alpha$ to make $\beta$, when they are reversible, is also given.



## 1. Introduction

At the beginning of the 20th century, Henry E. Dudeney [6] proposed the 'Haberdasher's Problem' of dissecting a regular triangle into four pieces which can be rearranged to form a square (see Fig. 1.1). A pair of polygons $\alpha$ and $\beta$ is said to be *equi-decomposable* if $\alpha$ has a dissection into a finite number of pieces which can be rearranged to form $\beta$ (see [4,9] for equi-decomposability of polygons and [10,5] for equi-decomposability of polyhedra). Reversibility is a general case of Dudeney's hinged dissection (see [8,3]) and a specific case of equi-decomposability.

A three dimensional analogue of reversibility occurs in the fields of crystallography and biology. In crystallography, we often encounter a "parallelohedron" as a Voronoi cell of a mineral crystal. As a result of phase transition, the corresponding Voronoi cell changes its shape from one parallelohedron to another (see [7,11]). In biology, a volvox, a kind of green alga known as one of the most simple colonial (multicellular) organisms, reproduces itself by reversing its interior offspring and its surface. Thus it is important to study shape changes in general and reversions in particular.

We briefly introduce conditions for a pair of polygons to be reversible, based on the results found in [1,12,2]. We then give some new results, a decision algorithm and complete classification of reversible pairs. The algorithm and classification are computable in constant time.

Throughout this paper, we deal with only convex polygons unless we explicitly state that the polygon is not convex. In this paper, we dissect polygons along the edges of a tree-like structure that we refer to as a *dissection tree*.

A given pair of polygons $\alpha$ and $\beta$ is said to be *reversible* if $\alpha$ and $\beta$ have dissections into a common finite number of pieces along edges of dissection trees which can be rearranged to form $\beta$ and $\alpha$ respectively, under the following conditions:

(i) The whole perimeter of one polygon fits into the interior of the other without gaps or overlaps and
(ii) The dissection tree of either polygon does not include any vertex of that polygon.

* Corresponding author. Tel.: +81 3 3269 7200.
*E-mail addresses:* ja@jin-akiyama.com (J. Akiyama), daver@cs.queensu.ca (D. Rappaport), hwseong@hotmail.com (H. Seong).

http://dx.doi.org/10.1016/j.dam.2014.06.015

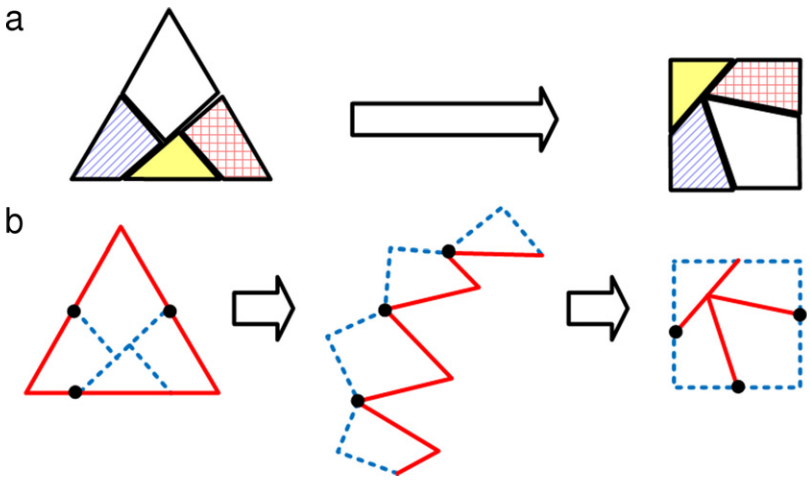


**Fig. 1.1.** (a) The answer to Dudeney's haberdasher's puzzle. (b) The corresponding hinged dissection.

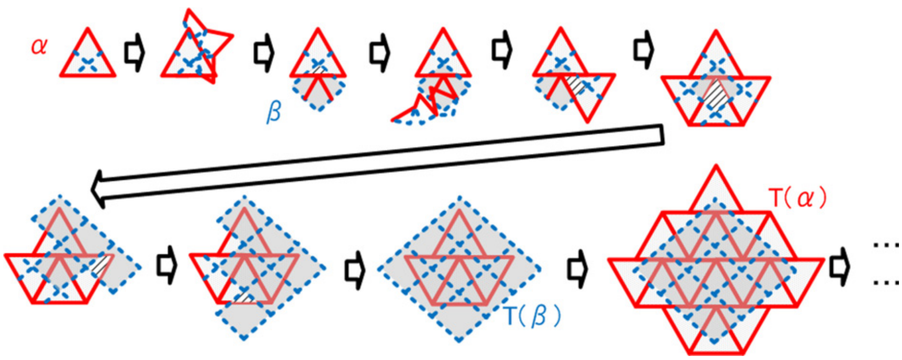


**Fig. 1.2.** The superimposition $T(\alpha, \beta)$ of $T(\alpha)$ and $T(\beta)$.

A polygon $\alpha$ is said to be *reversible* if there exists a polygon $\beta$ such that the pair $\alpha$ and $\beta$ is reversible. Note that the same term is used for a pair of polygons as well as a single polygon.

When a pair $\alpha$ and $\beta$ is reversible, we *hinge* the pieces of $\alpha$ ($\beta$) like a tree along the perimeter of $\alpha$ ($\beta$). It is always possible to hinge the $n$ pieces of $\alpha$ ($\beta$) by using point hinges at arbitrary $n - 1$ points among the $n$ points on the perimeter of $\alpha$ ($\beta$) and transform the hinged pieces to both $\alpha$ and $\beta$ continuously. Therefore we use models of hinged pieces without explicitly describing how to hinge and move the pieces.

When the 2-dimensional plane is tiled by congruent copies of a polygon $\alpha$ without gaps or overlaps, we say that $\alpha$ *tiles* the plane and denote the tiling by $\boldsymbol{T}(\boldsymbol{\alpha})$. Each copy of $\alpha$ is called a *cell* of the tiling $T(\alpha)$.

A tiling $T(\alpha)$ made by 180° rotations and translations only is called a $\mathbf{P_2}$-*tiling*.

**Theorem A** (*[1, Theorem 3.2]*)**.** *If a pair $\alpha$ and $\beta$ is reversible, each of $\alpha$ and $\beta$ tiles the plane by* 180° *rotations and translations only.*

It follows from Theorem A that if a pair $\alpha$ and $\beta$ is reversible, two tilings $T(\alpha)$ and $T(\beta)$ are induced on the same plane by a sequence of reversions between $\alpha$ and $\beta$. The *superimposition* of $T(\alpha)$ and $T(\beta)$ on the same plane obtained in this manner is denoted by $\boldsymbol{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ (see Fig. 1.2).

A polygon which tiles the plane in $P_2$-manner is called a *quasi-parallelogon*. Denote by **QP** the set of all quasi-parallelogons. Note that every reversible polygon belongs to the set QP by Theorem A.

**Proposition A** (*[12]*)**.** *Each reversible polygon belongs to one of seven* classes, *namely,*

(i) *the set of all triangles;*
(ii) *the set of all convex quadrilaterals with no pairs of parallel sides;*
(iii) *the set of all trapezoids with exactly one pair of parallel sides;*
(iv) *the set of all parallelograms;*
(v) *the set of all* house pentagons, *each of whose elements is a convex pentagon with a pair of parallel sides with the same length;*