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Bounds on weak roman and 2-rainbow domination numbers

Mustapha Chellali [a,*], Teresa W. Haynes [b,c], Stephen T. Hedetniemi [d]

[a] LAMDA-RO Laboratory, Department of Mathematics, University of Blida, B.P. 270, Blida, Algeria
[b] Department of Mathematics, East Tennessee State University, Johnson City, TN 37614, USA
[c] Department of Mathematics, University of Johannesburg, Auckland Park, South Africa
[d] School of Computing, Clemson University, Clemson, SC 29634, USA

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ABSTRACT

We mainly study two related dominating functions, namely, the weak Roman and 2-rainbow dominating functions. We show that for all graphs, the weak Roman domination number is bounded above by the 2-rainbow domination number. We present bounds on the weak Roman domination number and the secure domination number in terms of the total domination number for specific families of graphs, and we show that the 2-rainbow domination number is bounded below by the total domination number for trees and for a subfamily of cactus graphs.

1. Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and the degree of v, denoted as $d_G(v)$, is the cardinality of its open neighborhood, that is, $d_G(v) = |N(v)|$. An *isolated vertex* is a vertex with degree zero. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. For a subset $S \subseteq V$, the subgraph induced by S in G is denoted by $G[S]$. The *diameter*, $\mathrm{diam}(G)$, of a graph G is the maximum distance over all pairs of vertices of G. We denote the *star* with one central vertex and k leaves as $K_{1,k}$ and the *double star* with exactly two adjacent central support vertices having p and q leaf neighbors, respectively, as $S_{p,q}$. An *isolate-free graph* is a graph with no isolated vertices. The star $K_{1,3}$ is called a *claw*, and a graph is *claw-free* if it has no $K_{1,3}$ as an induced subgraph.

A subset $S \subseteq V$ is a *dominating set* if every vertex in $V \setminus S$ has a neighbor in S, and S is an *independent dominating set* if it is both dominating and independent (i.e. no two vertices in S are adjacent). The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G, and the *independent domination number* $i(G)$ equals the minimum cardinality of an independent dominating set of G. A subset $S \subseteq V$ is a *total dominating set*, abbreviated *TDS*, if every vertex in V has a neighbor in S, that is, $N(v) \cap S \neq \emptyset$ for all $v \in V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a TDS of G, and a TDS of G with minimum cardinality is called a $\gamma_t(G)$-set. Total domination was introduced by Cockayne et al. in [10]. For more on total domination, the reader is referred to the excellent book by Henning and Yeo [21].

* Corresponding author.
E-mail addresses: m_chellali@yahoo.com, mchellali@hotmail.com (M. Chellali), haynes@etsu.edu (T.W. Haynes), hedet@clemson.edu (S.T. Hedetniemi).

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Let k be a positive integer. As introduced by Fink and Jacobson in [14,15], a subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G, if $|N(v) \cap D| \geq k$ for every $v \in V \setminus D$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. For more information on k-domination see the survey by Chellali et al. [7].

A function $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V(G)} f(u)$, and the *Roman domination number* $\gamma_R(G)$ is the minimum weight of such a function on G. Roman domination was introduced by Cockayne et al. in [11], and has since been studied in more than 67 papers, for example, in [19,27].

Let V_i be the set of vertices assigned the value i under a function f. Note that there is a 1-to-1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V, so we will write $f = (V_0, V_1, V_2)$. Weak Roman domination, a less restrictive version of Roman domination, was introduced by Henning and Hedetniemi in [20] as follows. Let $f = (V_0, V_1, V_2)$ be a function on a graph $G = (V, E)$. A vertex v with $f(v) = 0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex w with $f(w) > 0$. A function f is called a *weak Roman dominating function* (WRDF) if each vertex v with $f(v) = 0$ is adjacent to a vertex w with $f(w) > 0$, such that the function $f' = (V'_0, V'_1, V'_2)$ defined by $f'(v) = 1, f'(w) = f(w) - 1$, and $f'(u) = f(u)$ for all $u \in V \setminus \{v, w\}$, has no undefended vertex. The weight of a WRDF is the value $f(V) = \sum_{u \in V(G)} f(u)$, and the *weak Roman domination number* $\gamma_r(G)$ is the minimum weight of such a function on G. Further results on weak Roman domination can be found for example, in [4,12,23].

Another related function, the 2-rainbow dominating function, was introduced by Brešar et al. in [2] and studied recently in [25–27]. Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2\}$, that is, $f : V(G) \to \mathcal{P}(\{1, 2\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$, then f is called a *2-rainbow dominating function* (2RDF) of G. The weight of a 2RDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a 2-rainbow dominating function is called the *2-rainbow domination number* of G, denoted by $\gamma_{r2}(G)$. We say that a function f is a *$\gamma_{r2}(G)$-function* if it is a 2RDF and $w(f) = \gamma_{r2}(G)$.

Another type of domination related to weak Roman domination was introduced by Cockayne et al. in [13] and studied in [5,12]. A *secure dominating set* S of a graph G is a dominating set with the property that each vertex $u \in V \setminus S$ is adjacent to a vertex $v \in S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. Note that secure domination can be also defined by a function $f = (V_0, V_1)$ such that for every vertex $u \in V_0$, there exists $v \in N(u) \cap V_1$ and $(V_1 \setminus \{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of a secure dominating set (i.e. the minimum weight of a secure dominating function) is called the *secure domination number* $\gamma_s(G)$ of G.

A common theme among these domination parameters could be graph protection where a configuration of guards is considered. For instance, if the vertices of a dominating set (respectively, total dominating set) S represent guards, then each vertex in $V \setminus S$ (respectively, V) is "protected" by S. With a 2-dominating set S, every vertex in $V \setminus S$ is protected by at least two guards. Roman, weak Roman, and secure domination allow for "mobile" protection. In other words, a guard can protect the location at which it is located and move to a neighboring vertex to defend it. Roman domination was motivated by the Emperor Constantine's efforts to defend the Roman Empire from attackers [11]. In both Roman and weak Roman domination, up to two armies (guards) are allowed at each vertex, providing the ability for one army to leave its home base to protect a neighboring community while the home base retains an army to protect it. With a secure dominating set S, there is one guard at each vertex of S and at most one guard of S is allowed to move in order to protect a neighboring vertex in $V \setminus S$, such that each of S and the set of guards after the move is a dominating set.

Several relationships among these parameters have been established, some of which are summarized in the following theorem.

Theorem 1. *For every graph G,*

(i) (Hedetniemi *et al.* [19]) *if G has no isolated vertices, then* $\gamma_t(G) \leq \gamma_R(G)$.
(ii) (Henning and Hedetniemi [20]) $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.
(iii) (Wu and Xing [27]) $\gamma(G) \leq \gamma_{r2}(G) \leq \gamma_R(G) \leq 2\gamma(G)$.
(iv) (Cockayne *et al.* [13]) $\gamma_r(G) \leq \gamma_s(G)$.
(v) (Cockayne *et al.* [12]) *if G is claw-free, then* $\gamma_r(G) = \gamma_s(G)$.
(vi) (Chellali and Rad [8]; Fujita and Furuya [16]) $\gamma_R(G) \leq 3\gamma_{r2}(G)/2$.

In this paper, we extend the inequalities in Theorem 1 by presenting some new relationships between these and other parameters. In particular, in Section 2, we show that $\gamma_r(G) \leq \gamma_{r2}(G)$ and $\gamma_s(G) \leq \gamma_2(G)$.

We note that some of these parameters are incomparable in general and the difference between two of them can be arbitrarily large. Indeed, consider the connected graphs H_k and F_k, where H_k is obtained from $k(k \geq 2)$ copies of the star $K_{1,3}$ by adding $k - 1$ edges incident only with the support vertices to connect the graph, while F_k for $k \geq 1$ is obtained from $2k$ copies of the cycle C_6 by first identifying a vertex of each C_6 with a vertex of a path P_{2k} so that the resulting graph is connected and the C_6's are vertex disjoint, and then subdividing each of the $2k - 1$ edges of the path P_{2k} exactly once. We observe that $\gamma(H_k) = \gamma_t(H_k) = k$, $\gamma_r(H_k) = \gamma_R(H_k) = \gamma_{r2}(H_k) = 2k$, and $\gamma_2(H_k) = \gamma_s(H_k) = 3k$, while $\gamma(F_k) = 4k$, $\gamma_t(F_k) = 7k$, $\gamma_r(F_k) = 6k$, $\gamma_R(F_k) = \gamma_{r2}(F_k) = 8k$, and $\gamma_2(F_k) = \gamma_s(F_k) = 6k$. These relationships for general graphs without isolated vertices are summarized in Table 1. We use the symbol $\diamond$ to denote incomparable and ? to denote open problem.