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Panconnectivity and edge-pancyclicity of multidimensional torus networks[☆]

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ABSTRACT

In this paper, the (bi)panconnectivity and edge-(bi)pancyclicity of n-dimensional torus networks are investigated. An n-dimensional torus $T = T(k_1, k_2, \ldots, k_n)$ with diameter $m_0 = \sum_{i=1}^{n} \lfloor k_i/2 \rfloor$, where $k_i \geq 3$ for $i = 1, 2, \ldots, n$, is one of the most popular interconnection networks, the k-ary n-cube Q_n^k $(=T(k, k, \ldots, k))$ is its special class. For any two vertices u and v in T, we determine the set $\rho(u, v)$ of all lengths of (u, v)-paths in T by using path-shortening technique that can be used efficiently to construct the (u, v)-paths in the torus T. In particular, the following results are obtained: (1) The torus T is bipanconnected and edge-bipancyclic; (2) If some $k_j \geq 3$ is odd and the other $k_i \geq 4$ is even for every $i \neq j$, then the torus T is m_1-panconnected, where $m_1 = (k_j - 1)/2 + m_0$, and the bound m_1 is optimal; (3) If both $k_i \geq 3$ and $k_j \geq 3$ are odd, then the torus T is m_0-panconnected, and the bound m_0 is optimal. (4) If some k_j is odd, let k be the minimum over all odd k_i, then T is $(k+1)$-edge pancyclic and the bound $k+1$ is optimal if $T \neq Q_n^k$, and T is h-edge-pancyclic and the bound $h = \max\{k-1, 3\}$ is optimal otherwise. Our results strengthen and generalize existing results.

1. Introduction

One of the central issues in designing and evaluating an interconnection network is to study how well other networks can be embedded into this network. Linear arrays and rings, which are two fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication cost. In this regard, embedding of paths and/or cycles of various lengths in networks is important and has become the area of intense study. See a survey of Xu and Ma [25] and references therein.

Let G be a graph and $|V(G)|$ denote the number of vertices in G. Given two vertices u and v in G. The number of edges in a shortest (u, v)-path is called the distance between u and v in G, and denoted by $d(u, v)$. The diameter of a graph is the maximum of the distances $d(u, v)$ over all vertex pairs u and v in the graph. A graph G is said to be m-panconnected (respectively, bipanconnected) if for any two vertices u and v in G and for every integer l with $(d(u, v) \leq) m \leq l \leq |V(G)| - 1$ (respectively, $d(u, v) \leq l \leq |V(G)| - 1$ and $l - d(u, v)$ is even), there exists a (u, v)-path of length l in G. A graph G is said to be (bi)pancyclic if G contains a cycle of every (even) length from $g(G)$ to $|V(G)|$, where $g(G)$ denotes the length of a shortest cycle in G. A graph G is said to be h-edge-(bi)pancyclic if given any edge e, G contains a cycle passing through the edge e of every (even) length from h to $|V(G)|$. There is a large amount of literature on (bi)panconnectivity and (bi)pancyclicity of networks,

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see recent Refs. [3–17,19,22–26,28]. An n-dimensional torus $T = T(k_1, k_2, \ldots, k_n)$, where $k_i \geq 3$ for $i = 1, 2, \ldots, n$, is one of the most popular interconnection networks [1,13,20,21,27,28], the k-ary n-cube Q_n^k $(= T(k, k, \ldots, k))$ is its special class. The path and/or cycle embedding of Q_n^k has been widely studied [10–12,14,18,19,24], and the path and/or cycle embedding of T has been studied in [13,28].

In this paper, we investigate the (bi)panconnectivity and edge-(bi)pancyclicity of an n-dimensional torus. For any two vertices u and v in the torus $T = T(k_1, k_2, \ldots, k_n)$ with diameter $m_0 = \sum_{i=1}^{n} \lfloor k_i/2 \rfloor$, we determine the set $\rho(u, v)$ of all lengths of (u, v)-paths in T by using path-shortening technique that can be used efficiently to construct the (u, v)-paths in the torus T. In particular, we show that: The torus T is bipanconnected and edge-bipancyclic; and T is also $((k_j - 1)/2 + m_0)$-panconnected if some $k_j \geq 3$ is odd and the other $k_i \geq 4$ is even for $i \neq j$; and T is m_0-panconnected if both $k_i \geq 3$ and $k_j \geq 3$ are odd. And the two bounds are optimal. If some k_j is odd, let k be the minimum over all odd k_i, then T is $(k+1)$-edge pancyclic and the bound $k + 1$ is optimal if $T \neq Q_n^k$, and T is h-edge-pancyclic and the bound $h = \max\{k - 1, 3\}$ is optimal otherwise. Our results strengthen and generalize the results in [13,19,28].

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give some lemmas. In Section 4, we investigate the (bi)panconnectivity and edge-(bi)pancyclicity of the torus network. Conclusions are presented in Section 5.

2. Preliminaries

In this paper, we follow [2] for terminology and notation of graph theory. A graph $G = (V, E)$ means an undirected graph without loops and multiple edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of the graph G. We use $P = (v_0, v_1, \ldots, v_k)$ to denote a path with $k + 1$ vertices $v_0, v_1, \ldots, v_k$ and k edges (v_i, v_{i+1}) for $i = 0, 1, \ldots, k - 1$, where two vertices v_0 and v_k are called its end-vertices, and P is also called a (v_0, v_k)-path. If $P = (v_0, \ldots, v_i, \ldots, v_j, \ldots, v_k)$ is a path, then $(v_i, \ldots, v_j)$ is called a subpath of the path P and denoted by $P[v_i, v_j]$. We use $C = v_1 v_2 \ldots v_k$, where $v_k = v_1$, to denote a cycle with $k(\geq 3)$ edges. The number of the edges in a path P (respectively, cycle C) is called its length and denoted by $l(P)$ (respectively, $l(C)$). Cartesian product of two graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$, in which a vertex (u, v) is adjacent to a vertex (u', v') if and only if either $u = u'$ in G and v is adjacent to v' in H, or $v = v'$ in H and u is adjacent to u' in G. A path (respectively, cycle) containing all vertices of a graph is called its Hamiltonian path (respectively, Hamiltonian cycle). A path containing all but one vertices of a graph is called its an almost Hamiltonian path. If P and P' are two paths such that their only one common vertex is an end-vertex of both P and P', then the path induced by $E(P) \cup E(P')$ is denoted by $P \cup P'$.

Let C_{k_i} denote a cycle with $k_i(\geq 3)$ edges for $i = 1, 2, \ldots, n$. Cartesian product $C_{k_1} \times C_{k_2} \times \cdots \times C_{k_n}$ is called a (n-dimensional) torus and denoted by $T(k_1, k_2, \ldots, k_n)$. If $k_1 = k_2 = \cdots = k_n = k \geq 3$, then a torus $T(k, k, \ldots, k)$ is called a k-ary n-cube and denoted by Q_n^k. It is clear that a torus has vertex-symmetry and has diameter $m_0 = \sum_{i=1}^{n} \lfloor k_i/2 \rfloor$. If $n \geq 2$, then $T(k_1, k_2, \ldots, k_n) = C_{k_1} \times T(k_2, \ldots, k_n) = C_{k_j} \times T(k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_n)$. This fact will be used to prove the following theorems.

By the definition of Cartesian product, a torus can also be defined as follows. Let $k_i \geq 3$ for $i = 1, 2, \ldots, n$. A torus $T(k_1, k_2, \ldots, k_n)$ is a graph with $N = \prod_{i=1}^{n} k_i$ vertices, its any vertex v can be denoted by an n-tuple $v = (x_1, x_2, \ldots, x_n)$, where $0 \leq x_i \leq k_i - 1$ for $i = 1, 2, \ldots, n$, and the vertex v is adjacent to exactly $2n$ vertices $(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$ and $(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n)$, where $x_i + 1$ and $x_i - 1$ are taken modulo k_i for $i = 1, 2, \ldots, n$. For any i, an edge between two vertices $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ and $(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$ (or $(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n)$) is called an edge of dimension i, and the edge between two vertices $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ and $(x_1, \ldots, x_{i-1}, k_i - 1, x_{i+1}, \ldots, x_n)$ is called the "wrap-around" edge of dimension i.

The following definition is similar to that given by Stewart and Xiang in [19].

Definition 1. Let $P_0 = (v_0, v_1, \ldots, v_i, v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{n-1}, v_n)$ be a (v_0, v_n)-path of length $n(\geq 3)$ in a graph G. If (v_i, v_{i+3}) is an edge in G, where $0 \leq i \leq n - 3$, then $P_1 = (v_0, v_1, \ldots, v_i, v_{i+3}, \ldots, v_{n-1}, v_n)$ is a (v_0, v_n)-path of length $n - 2$ in the graph G. We say that the path P_0 can be shortened to the path P_1. If for $j = 0, 1, \ldots, s - 1$, the path P_j of length $n - 2j$ can be shortened to a path P_{j+1} of length $n - 2j - 2$, then we say that the path P_0 can be progressively shortened to a path P_s.

For example, in the following ladder graph G with $4k$ vertices, assume that P is the (u_0, v_0)-path with $4k$ vertices and Q is the (u_0, u_{2k-1})-path with $4k$ vertices. Then, by progressively shortening P (respectively, Q) we obtain the edge (u_0, v_0) (respectively, the shortest (u_0, u_{2k-1})-path). See Fig. 1.

3. Some lemmas

We give the following three lemmas.

Lemma 1. *Let $u = (0, 0, \ldots, 0)$ and $v = (d_1, d_2, \ldots, d_n)$ be two vertices in a torus $T = T(k_1, k_2, \ldots, k_n)$ with the distance $d(u, v) = \sum_{i=1}^{n} d_i \geq 1$, where $0 \leq d_i \leq k_i/2$ for $i = 1, 2, \ldots, n$, and some $k_j \geq 3$ is odd. And let $r = r(u, v) = \min\{k_i - 2d_i \mid k_i \text{ is odd}, 1 \leq i \leq n\}$, then $r(\geq 1)$ is odd. Assume that P is a (u, v)-path in the torus T of length $l(P) = d(u, v) + t$, where t is odd. Then $t \geq r$.*