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Hitting sets online and unique-max coloring[☆]

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ABSTRACT

We consider the problem of hitting sets online. The hypergraph is known in advance, and the hyperedges to be stabbed are input one-by-one in an online fashion. The online algorithm must stab each hyperedge upon arrival. The best known competitive ratio for hitting sets online by Alon et al. (2009) is $O(\log n \cdot \log m)$ for general hypergraphs, where n and m denote the number of vertices and the number of hyperedges, respectively.

In this paper we provide the following results:

1. We consider hypergraphs in which the union of two intersecting hyperedges is also a hyperedge. Our main result for such hypergraphs is as follows: We consider a recently studied hypergraph coloring notion referred to as "unique-maximum coloring" and show that the competitive ratio of the online hitting set problem is at most the unique-maximum chromatic number and at least this number minus one.

2. Given a graph $G = (V, E)$, let $H = (V, R)$ denote the hypergraph whose hyperedges are subsets $U \subseteq V$ such that the induced subgraph $G[U]$ is connected. We establish a new connection between the best competitive ratio for the online hitting set problem in H and a well studied graph coloring notion referred to as the "vertex ranking number" of G. This connection states that these two parameters are equal. Moreover, this equivalence is constructive. As a corollary, we obtain optimal online hitting set algorithms for many such hypergraphs including those realized by planar graphs, graphs with bounded tree width, trees, etc. This improves the best previously known general bound of Alon et al. (2009).

3. We also consider two geometrically defined hypergraphs. The first one is defined by subsets of a given set of n points in the Euclidean plane that are induced by half-planes. The second hypergraph is defined by subsets of a given set of n points in the plane induced by unit discs. For these hypergraphs, the competitive ratio obtained by Alon et al. is $O(\log^2 n)$. For both cases, we introduce an algorithm with $O(\log n)$-competitive ratio. We also show that any online algorithm for both settings has a competitive ratio of $\Omega(\log n)$, and hence our algorithms are optimal.

1. Introduction

In the minimum hitting set problem, we are given a hypergraph (X, R), where X is the ground set of vertices and R is a set of hyperedges. The goal is to find a "small" cardinality subset $S \subseteq X$ such that every hyperedge is stabbed by S, namely, every hyperedge has a nonempty intersection with S.

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The minimum hitting set problem is a classical NP-hard problem [15], and remains hard even for geometrically induced hypergraphs (see [12] for several references). A sharp logarithmic threshold for hardness of approximation was proved by Feige [11] (see also [20]). On the other hand, the greedy algorithm achieves a logarithmic approximation ratio [14,19,7]. Better approximation ratios have been obtained for several geometrically induced hypergraphs using specific properties of the underlying hypergraphs [12,17,2]. Other improved approximation ratios are obtained using the theory of VC-dimension and ε-nets [4,10,8]. Much less is known about online versions of the hitting set problem.

In this paper, we consider an online setting in which the hypergraph is given in the beginning, but the hyperedges that need to be stabbed are introduced one by one. Upon arrival of a new hyperedge, the online algorithm may add vertices (from X) to the hitting set so that the hitting set also stabs the new hyperedge. However, the online algorithm may not remove vertices from the hitting set. We use the competitive ratio for our analysis, a classical measure for the performance of online algorithms [23,3].

Alon et al. [1] considered the online set-cover problem for arbitrary hypergraphs. In their setting, the hypergraph is known in advance, but the vertices are introduced one by one. Upon arrival of an uncovered vertex, the online algorithm must choose a hyperedge that covers the vertex. Hence, by interchanging the roles of hyperedges and vertices, the online set-cover problem and the online hitting-set problems are equivalent. The online hitting-set algorithm presented by Alon et al. [1] achieves a competitive ratio of $O(\log n \log m)$ where n and m are the number of vertices and the number of hyperedges respectively. Note that if $m \geq 2^{n/\log n}$, the analysis of the online algorithm only guarantees that the competitive ratio is $O(n)$; a trivial bound because one may simply choose one vertex to stab each hyperedge.

Unique-maximum coloring. We need a recently studied hypergraph coloring notion. Let $H = (X, R)$ be a hypergraph. A coloring $c : X \to \{1, \ldots, k\}$ is a *unique-max coloring* or *UM-coloring* H if, for each hyperedge $r \in R$, exactly one vertex in r is colored by the color $\max_{x \in r} c(x)$. Let $\chi_{um}(H)$ denote the least integer k for which H admits a UM-coloring with k colors. The notion of UM-coloring was first introduced and studied in [24,9]. This notion attracted many researchers and has been the focus of many research papers both in the computer science and mathematics communities. Such colorings arise in the context of frequency assignment to cellular antennae, in RFID protocols, and several other fields. (cf., [25]).

Let $G = (V, E)$ be a simple graph. An *ordered coloring* (also a *vertex ranking*) of G is a coloring of the vertices $\chi : V \to \{1, \ldots, k\}$ such that whenever two vertices u and v have the same color i then every simple path between u and v contains a vertex with color greater than i. Such a coloring has been studied before and has several applications. It was studied in the context of VLSI design [22] and in the context of parallel Cholesky factorization of matrices [18]. The vertex ranking problem is also interesting for the Operations Research community. It has applications in planning efficient assembly of products in manufacturing systems [13]. See also [16,21].

The vertex ranking coloring is yet another special form of UM-coloring. Given a graph G, consider the hypergraph $H = (V, E')$ where a subset $V' \subseteq V$ is a hyperedge in E' if and only if V' is the set of vertices in some simple path of G. It is easily observed that an ordered coloring of G is equivalent to a UM-coloring of H.

Relation between unique-maximum coloring and the competitive ratio. We consider the competitive ratio for the hitting set problem as a property of the underlying hypergraph. Namely, the competitive ratio of a hypergraph $H = (X, R)$ is the competitive ratio of the best deterministic online algorithm for the hitting set problem for H. We say that a hypergraph is union-closed if the union of two intersecting hyperedges is always a hyperedge. Our main result (Theorem 7) shows a new connection between the competitive ratio of a union-closed hypergraph H and the minimum number of colors required to color H in a unique-max coloring. In fact, we present "black box" reductions that construct an online hitting set algorithm from a unique-max coloring, and vice-versa.

Applications. Three applications of the main result are presented. The first application is motivated by the following setting. Consider a communication network $G = (V, E)$. This network is supposed to serve requests for virtual private networks (VPNs). Each VPN request is a subset of vertices that induces a connected subgraph in the network, and requests for VPNs arrive online. For each VPN, we need to assign a server (among the nodes in the VPN) that serves the nodes of the VPN. Since setting up a server is expensive, the goal is to select as few servers as possible.

This application can be abstracted by considering the hypergraph H whose hyperedges are all subsets of vertices of a given graph G that induce a connected subgraph. This hypergraph captures the online problem in which the adversary chooses subsets $V' \subseteq V$ such that the induced subgraph $G[V']$ is connected, and the algorithm must stab the subgraphs. A direct consequence of the observation that every unique-max coloring of H is a vertex ranking of G implies that the competitive ratio of H equals the vertex ranking number of G. This application leads to improved optimal competitive ratios for graphs that admit (hereditary) small balanced separators (see Table 1).

Two more classes of hypergraphs are obtained geometrically as follows. In both settings we are given a set X of n points in the plane. In one hypergraph, the hyperedges are intersections of X with half planes. In the other hypergraph, the hyperedges are intersections of X with unit discs. Although these hypergraphs are not union-closed, we present an online algorithm for the hitting set problem for points in the plane and unit discs (or half-planes) with an optimal competitive ratio of $O(\log n)$. The competitive ratio of this algorithm improves the $O(\log^2 n)$-competitive ratio of Alon et al. by a logarithmic factor.

An application for points and unit discs is the selection of access points or base stations in a wireless network. The points model base stations and the disc centers model clients. The reception range of each client is a disc, and the algorithm has to select a base station that serves a new uncovered client. The goal is to select as few base stations as possible.