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Labeling amalgamations of Cartesian products of complete graphs with a condition at distance two

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ABSTRACT

The spectrum allocation problem in wireless communications can be modeled through vertex labelings of a graph, wherein each vertex represents a transmitter and edges connect vertices whose corresponding transmitters are operating in close proximity. One well-known model is the $L(2, 1)$-labeling of a graph G in which a function f maps the vertices of G to the nonnegative integers such that if vertices x and y are adjacent, then $|f(x)-f(y)| \geq 2$, and if x and y are at distance two, then $|f(x)-f(y)| \geq 1$. The λ-number of G is the minimum span over all $L(2, 1)$-labelings of G. Informally, an amalgamation of two graphs G_1 and G_2 along a fixed graph G_0 is the simple graph obtained by identifying the vertices of two induced subgraphs isomorphic to G_0, one in G_1 and the other in G_2. In this work, we supply a tight upper bound for the λ-number of amalgamations of several Cartesian products of complete graphs along a complete graph and find the exact λ-numbers for certain infinite subclasses of amalgamations of this form. A surprising relationship between the former upper bound and the minimum makespan scheduling problem is highlighted.

1. Introduction

An $L(2, 1)$-*labeling* of a graph G is an assignment of non-negative integers to its vertices such that adjacent vertices must receive integers at least two apart, and vertices at distance two must receive integers at least one apart. The study of $L(2, 1)$-labelings and their variations was motivated by the channel assignment problem [10] and has generated a vast literature since these labelings were introduced in 1992 [9]. We refer the reader to the surveys in [2,17] and a sample of the most recent works in the field in [3,4,12–16].

A k-*labeling* of a graph G is an $L(2, 1)$-labeling that uses labels in the set $\{0, 1, \ldots, k\}$. The minimum k so that G has a k-labeling is called the λ-*number* of G and will be denoted by $\lambda(G)$. The long-standing conjecture in the field states that $\lambda(G) \leq \Delta^2(G)$, where $\Delta(G)$ denotes the maximum degree of G [9]. This conjecture holds for graphs with $\Delta(G)$ larger than approximately 10^{69} [11] and for graphs with at most $(\lfloor \Delta(G)/2 \rfloor + 1)(\Delta^2(G) - \Delta(G) + 1) - 1$ vertices [4]. The best known general upper bound is $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$ [8]. Even though the general problem of determining $\lambda(G)$ is NP-hard [7], several bounds and exact λ-numbers for different families of graphs are known. One of these families is the class of amalgamations of graphs studied in [1].

Definition 1.1. Let $G_1, G_2, \ldots, G_p$ be $p \geq 2$ pairwise disjoint graphs each containing a fixed induced subgraph isomorphic to a graph G_0. The *amalgamation* of $G_1, G_2, \ldots, G_p$ *along* G_0 is the simple graph $G = Amalg(G_0; G_1, G_2, \ldots, G_p)$ obtained by

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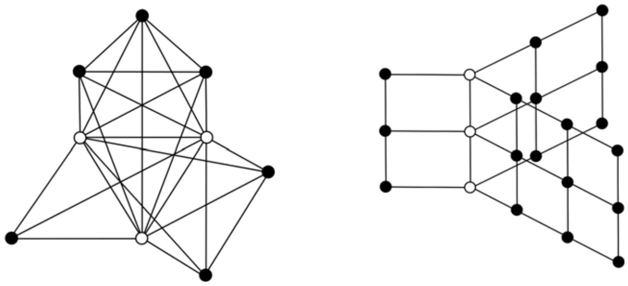
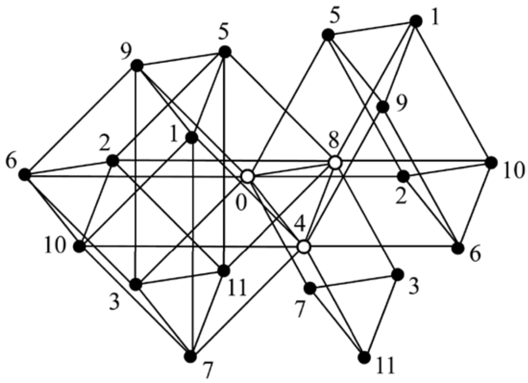


Fig. 1.1. $Amalg(K_3; K_6, K_5, K_4)$ and $Amalg(P_3; P_3 \square P_4, P_3 \square P_3, P_3 \square P_2)$, respectively.



0	9	6	3
4	1	10	7
8	5	2	11

0	5	2
4	9	6
8	1	10

0	7
4	11
8	3

Fig. 1.2. $L(2, 1)$-labeling of $Amalg(K_3; K_3 \square K_4, K_3 \square K_3, K_3 \square K_2)$ and the corresponding matrix representation.

identifying $G_1, G_2, \ldots, G_p$ at the vertices in the fixed subgraphs isomorphic to G_0 in each $G_1, G_2, \ldots, G_p$, respectively. G_0 is called the *spine* and G_k is called *page k* of G for $k = 1, 2, \ldots, p$.

In [1], general upper bounds for the λ-number of the amalgamation of graphs were established by determining the exact λ-number of the amalgamation of complete graphs along a complete graph. They also provided the exact λ-numbers of the amalgamation of rectangular grids along a certain path, or more specifically, of the Cartesian product of a path and a star with spokes of arbitrary lengths. This focus on the Cartesian product in the context of amalgamations motivated us to investigate the λ-number of the amalgamation of Cartesian products of complete graphs along a complete graph.

Definition 1.2. The *Cartesian product* of two disjoint graphs G and H, denoted by $G \square H$, is defined as the graph with vertex set given by the Cartesian product of the vertex set of G and the vertex set of H, where two vertices (u, v) and (w, z) are adjacent if and only if either $[u, w$ are adjacent in G and $v = z]$ or $[v, z$ are adjacent in H and $u = w]$.

Throughout, $p, n_0, n_1, \ldots, n_p$, are all integers greater than or equal to 2, unless otherwise noted. We will study the amalgamation K of Cartesian products of complete graphs along a complete graph, more specifically, $K = Amalg(K_{n_0}; K_{n_0} \square K_{n_1}, K_{n_0} \square K_{n_2}, \ldots, K_{n_0} \square K_{n_p})$ where K_{n_k} is the complete graph on n_k vertices for $k = 0, 1, \ldots, p$. For a fixed $k = 1, 2, \ldots, p$, the vertices in page k, that is, the vertices in $K_{n_0} \square K_{n_k}$, can be organized in an array format where each vertex will be represented by an ordered triple (i, j, k) with $i = 0, 1, \ldots, n_0 - 1$, and $j = 0, 1, \ldots, n_k - 1$ so that two vertices are adjacent if their triple representations satisfy exactly one of the following conditions:

(i) Both triples agree on the first and last coordinate, respectively.
(ii) Both triples agree on the second and last coordinate, respectively.

The subgraph induced by the vertices in the same row of this array is isomorphic to K_{n_k} and the subgraph induced by the vertices in the same column is isomorphic to K_{n_0}. Furthermore, for a fixed i, the vertices $(i, 0, k)$ for $k = 1, 2, \ldots, p$, represent the same vertex s_i in the spine K_{n_0}. For convenience, $L(2, 1)$-labelings of K will be represented by the n_0-by-n_k matrices, $k = 1, 2, \ldots, p$, where the entry on the ith row, jth column of the kth matrix will be the label of vertex (i, j, k); observe that all the 0th columns of these p matrices must be the same as they contain the labels for the spine.

To illustrate the different amalgamations mentioned in this section, we provide three examples: in Fig. 1.1, an amalgamation of complete graphs along a complete graph (on the left), and an amalgamation of rectangular grids along a path (on the right); in Fig. 1.2, an amalgamation of Cartesian products of complete graphs along a complete graph with an $L(2, 1)$-labeling (on the left) and the corresponding matrix representation (on the right). In each subfigure, the vertices in the spine are in white.