Discrete Applied Mathematics 178 (2014) 109–120

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Linear and cyclic distance-three labellings of trees

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ARTICLE INFO

Article history:
Received 10 September 2013
Received in revised form 3 June 2014
Accepted 5 June 2014
Available online 23 June 2014

Keywords:
Channel assignment
Frequency assignment
Distance three labelling
Cyclic labelling
λ-number
σ-number
Tree
Complete m-ary tree

ABSTRACT

Given a finite or infinite graph G and positive integers ℓ, h_1, h_2, h_3, an $L(h_1, h_2, h_3)$-labelling of G with span ℓ is a mapping $f : V(G) \to \{0, 1, 2, \ldots, \ell\}$ such that, for $i = 1, 2, 3$ and any $u, v \in V(G)$ at distance i in G, $|f(u) - f(v)| \geq h_i$. A $C(h_1, h_2, h_3)$-labelling of G with span ℓ is defined similarly by requiring $|f(u) - f(v)|_\ell \geq h_i$ instead, where $|x|_\ell = \min\{|x|, \ell - |x|\}$. The minimum span of an $L(h_1, h_2, h_3)$-labelling, or a $C(h_1, h_2, h_3)$-labelling, of G is denoted by $\lambda_{h_1,h_2,h_3}(G)$, or $\sigma_{h_1,h_2,h_3}(G)$, respectively. Two related invariants, $\lambda^*_{h_1,h_2,h_3}(G)$ and $\sigma^*_{h_1,h_2,h_3}(G)$, are defined similarly by requiring further that for every vertex u there exists an interval I_u mod $(\ell + 1)$ or mod ℓ, respectively, such that the neighbours of u are assigned labels from I_u and $I_v \cap I_w = \emptyset$ for every edge vw of G. A recent result asserts that the $L(2, 1, 1)$-labelling problem is NP-complete even for the class of trees. In this paper we study the $L(h, p, p)$ and $C(h, p, p)$ labelling problems for finite or infinite trees T with finite maximum degree, where $h \geq p \geq 1$ are integers. We give sharp bounds on $\lambda_{h,p,p}(T)$, $\lambda^*_{h,p,p}(T)$, $\sigma_{h,1,1}(T)$ and $\sigma^*_{h,1,1}(T)$, together with linear time approximation algorithms for the $L(h, p, p)$-labelling and the $C(h, 1, 1)$-labelling problems for finite trees. We obtain the precise values of these four invariants for complete m-ary trees with height at least 4, the infinite complete m-ary tree, and the infinite $(m + 1)$-regular tree and its finite subtrees induced by vertices up to a given level. We give sharp bounds on $\sigma_{h,p,p}(T)$ and $\sigma^*_{h,p,p}(T)$ for trees with maximum degree $\Delta \leq h/p$, and as a special case we obtain that $\sigma_{h,1,1}(T) = \sigma^*_{h,1,1}(T) = 2h + \Delta - 1$ for any tree T with $\Delta \leq h$.

1. Introduction

A. Motivation. In a radio communication network, a channel or a set of channels is required [17] to assign to each transmitter such that the bandwidth used is minimized while interference between transmitters of geographical proximity is maintained at an acceptable level. From a combinatorial point of view, this fundamental problem is essentially an optimal labelling problem for the corresponding interference graph, which is defined to have transmitters as its vertices such that two vertices are adjacent if and only if the corresponding transmitters are geographically close to each other. In the case when each transmitter requires only one channel, we seek an assignment of a nonnegative integer (label) to each vertex such that for i from 1 to some given integer d, whenever two vertices are distance i apart in the graph, the difference (in absolute value) between their labels is no less than a given separation. The existence of such an assignment is not a problem if sufficiently many channels are provided. However, since bandwidth is limited and costly, a major concern is to find the minimum span required among such channel assignments.

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http://dx.doi.org/10.1016/j.dam.2014.06.003
0166-218X/

B. Linear and cyclic labellings. The problem above can be modelled as follows. Given a finite or infinite undirected graph $G = (V(G), E(G))$ and a sequence of nonnegative integers $h_1, \ldots, h_d$, an $L(h_1, \ldots, h_d)$-*labelling* of G with *span* ℓ is a mapping $f : V(G) \to \{0, 1, 2, \ldots, \ell\}$ such that, for $i = 1, 2, \ldots, d$ and any $u, v \in V(G)$ with $d(u, v) = i$,

$$|f(u) - f(v)| \geq h_i, \tag{1}$$

where ℓ is a positive integer and $d(u, v)$ denotes the distance in G between u and v. (In this paper an *infinite graph* means a graph with countably infinitely many vertices. A graph is meant a finite graph unless stated otherwise.) In practical terms, the *label* of u under f, $f(u)$, is the channel assigned to the transmitter corresponding to u. Without loss of generality we may always assume $\min_{v \in V(G)} f(v) = 0$. The $\lambda_{h_1,\ldots,h_d}$-*number* of G, denoted $\lambda_{h_1,\ldots,h_d}(G)$, is defined [16,17] to be the minimum span of an $L(h_1, \ldots, h_d)$-labelling of G. Equivalently, $\lambda_{h_1,\ldots,h_d}(G) = \min_f \operatorname{span}(f)$, with minimum over all $L(h_1, \ldots, h_d)$-labellings of G, where $\operatorname{span}(f) = \max_{v \in V(G)} f(v)$. In practice this parameter measures [17] the minimum bandwidth required by the radio communication network under constraints (1).

The $L(h_1, \ldots, h_d)$-labelling problem above is a linear model in the sense that the metric involved is the ℓ_1-metric. Its cyclic version was studied in [24] with a focus on small d. A $C(h_1, \ldots, h_d)$-*labelling* of a finite or infinite graph G with *span* ℓ is a mapping $f : V(G) \to \{0, 1, 2, \ldots, \ell - 1\}$ such that, for $i = 1, 2, \ldots, d$ and any $u, v \in V(G)$ with $d(u, v) = i$,

$$|f(u) - f(v)|_\ell \geq h_i, \tag{2}$$

where $|x - y|_\ell := \min\{|x - y|, \ell - |x - y|\}$ is the ℓ-*cyclic distance* between integers x and y. A $C(h_1, \ldots, h_d)$-labelling of G with span ℓ exists for sufficiently large ℓ. Define the $\sigma_{h_1,\ldots,h_d}$-*number* of G, denoted $\sigma_{h_1,\ldots,h_d}(G)$, to be the minimum integer ℓ such that G admits a $C(h_1, \ldots, h_d)$-labelling with span ℓ. Note that $\sigma_{h_1,\ldots,h_d}(G)$ thus defined agrees with $\sigma(G; h_1, \ldots, h_d)$ in [24] and $c_{h_1,\ldots,h_d}(G)$ in [22], but is larger by one than $\sigma(G; h_1, \ldots, h_d)$ in [8]. As observed in [11,24], this cyclic version allows the assignment of a set of channels $f(u), f(u) + \ell, f(u) + 2\ell, \ldots$ to each transmitter u. The possibility of providing such multiple coverage is important [24] in large communication systems that serve many customers simultaneously.

We will refer to (1) and (2) as the $L(h_1, \ldots, h_d)$ *conditions* and the $C(h_1, \ldots, h_d)$ *conditions* mod ℓ, respectively.

C. Literature review. The linear and cyclic labelling problems above are interesting in both theory and practical applications, and as such they have been studied extensively over many years, especially in the case when $d = 2$. In the simplest looking case when $d = 1$, the $L(h_1)$-labelling problem becomes the classic vertex-colouring problem (which is important and difficult already) because $\lambda_{h_1}(G) = h_1(\chi(G) - 1)$, where $\chi(G)$ is the chromatic number of G. Another interesting and important special case is that $\lambda_{1,\ldots,1}(G) = \chi(G^d) - 1$, where G^d is the d-th power of G defined to have vertex set $V(G)$ such that two vertices are adjacent if and only if their distance in G is at most d. In the case when $d = 2$, many interesting results on λ_{h_1,h_2} have been obtained by various researchers for many families of finite graphs, especially when $(h_1, h_2) = (2, 1)$; see e.g. [6–8,12,16,26,25,2] for an extensive bibliography. Griggs and Yeh [16] conjectured that $\lambda_{2,1}(G) \leq \Delta^2$ for any graph G with maximum degree $\Delta \geq 2$. This has been confirmed for several classes of graphs, including chordal graphs [23], generalized Petersen graphs [13], Hamiltonian graphs with $\Delta \leq 3$ [19], etc. Improving earlier results [6,16], Goncalves [15] proved that $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$ for any graph G with $\Delta \geq 2$. A recent breakthrough in this direction by Havet, Reed and Sereni [18] asserts that for any $h \geq 1$ there exists a constant $\Delta(h)$ such that every graph with maximum degree $\Delta \geq \Delta(h)$ satisfies $\lambda_{h,1} \leq \Delta^2$. In particular, the Griggs–Yeh conjecture is true for any graph with sufficiently large maximum degree.

In [16] it was proved that $\lambda_{2,1}(T) = \Delta + 1$ or $\Delta + 2$ for any tree T. A polynomial time algorithm for determining $\lambda_{2,1}(T)$ was given in [6], while in general it was conjectured [10] that the problem of determining λ_{h_1,h_2} (where $h_1 > h_2 \geq 1$) for trees is NP-hard. In [5] it was proved that $\Delta + h - 1 \leq \lambda_{h,1}(T) \leq \min\{\Delta + 2h - 2, 2\Delta + h - 2\}$ with both bounds attainable. In [14], for $h_1 \geq h_2$ the λ_{h_1,h_2}-number was derived for infinite regular trees. In [4], for $h_1 < h_2$ the smallest integer λ such that every tree of maximum degree $\Delta \geq 2$ admits an $L(h_1, h_2)$-labelling of span at most λ was studied.

In [24] the $C(h_1, \ldots, h_d)$ labelling problem was studied for infinite triangular lattice, infinite square lattice and infinite line lattices with a focus on the cases $d = 2, 3$. In [22] it was proved that, for a graph G with n vertices, if its complement G^c is Hamiltonian, then $\sigma_{2,1}(G) \leq n$; otherwise $\sigma_{2,1}(G) = n + p_v(G^c)$, where $p_v(G^c)$ is the smallest number of vertex-disjoint paths covering $V(G^c)$. In [8] it was proved that for a Hamming graph $K_{n_1} \Box \cdots \Box K_{n_t}$ (where $n_1 \geq \cdots \geq n_t$ and $\Box$ denotes the Cartesian product), if n_1 is sufficiently large relative to $n_2, \ldots, n_t$, then $\lambda_{2,1}$, $\lambda_{1,1}$ and their 'no-hole' counterparts are all equal to $n_1 n_2 - 1$, and $\sigma_{2,1}$, $\sigma_{1,1}$ and their 'no-hole' counterparts are all equal to $n_1 n_2$.

Relatively few results were known for the linear and cyclic labelling problems when $d = 3$. In [21] King, Ras and Zhou obtained sharp bounds on $\lambda_{h,1,1}$ for trees. They asked whether for fixed $h \geq 2$ the $L(h, 1, 1)$-labelling problem for trees can be solved in polynomial time. Answering this question, in [9] it was proved among others that the $L(2, 1, 1)$-labelling problem is NP-complete for trees. This indicates that even for trees the $L(h_1, h_2, h_3)$-labelling problem is difficult in general. In [27] the third author of the present paper obtained tight upper bounds on $\lambda_{h_1,h_2,h_3}(Q_n)$ by using a group-theoretic approach, where Q_n is the n-dimensional cube. In [3] a linear time approximation algorithm to $L(h, 1, 1)$-label an outerplanar graph was given, using span $3\Delta + 8$ when $h = 1$ and $\Delta \geq 6$, and $3\Delta + 2h + 6$ when $h \geq 2$ and $\Delta \geq 4h + 7$. In [1] an $O(d^2 h_1 n)$-time approximation algorithm was given for the $L(h_1, \ldots, h_d)$-labelling problem for trees, with performance ratio depending on h_i and $\lambda(T^i)$ for $i = 1, \ldots, d$. In a recent paper [20] the $\lambda_{h_1,h_2,1}$-number was determined for the direct product of K_2 and two other complete graphs under various conditions on h_1 and h_2.

D. Elegant labellings. Noting that the labellings producing the upper bounds in [9,21] assign an interval to the neighbourhood of each vertex, the following notion was introduced in [9]. Let f be an $L(h_1, \ldots, h_d)$ or $C(h_1, \ldots, h_d)$-labelling of G with