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Generalizations of bounds on the index of convergence to weighted digraphs

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ABSTRACT

We study sequences of optimal walks of a growing length in weighted digraphs, or equivalently, sequences of entries of max-algebraic matrix powers with growing exponents. It is known that these sequences are eventually periodic when the digraphs are strongly connected. The transient of such periodicity depends, in general, both on the size of digraph and on the magnitude of the weights. In this paper, we show that some bounds on the indices of periodicity of (unweighted) digraphs, such as the bounds of Wielandt, Dulmage–Mendelsohn, Schwarz, Kim and Gregory–Kirkland–Pullman, apply to the weights of optimal walks when one of their ends is a critical node.

1. Introduction

We show that six known bounds for the index of convergence (transient of periodicity) of an unweighted digraph also apply to weighted digraphs, namely, to transients of rows and columns with critical indices.

The origin of the first of these known bounds lies in Wielandt's well-known paper [25] where an upper bound for the transient of a primitive nonnegative matrix was asserted without proof.[1] Dulmage and Mendelsohn [10] provided a proof of this result by interpreting it in terms of digraphs and they sharpened the result by using as additional information in the hypotheses the length of the smallest cycle of the digraph.[2] Schwarz [18] generalized Wielandt's result to apply to all strongly connected digraphs by using Wielandt's bound for the cyclicity classes of the digraph, see also Shao and Li [23]. Kim's [14] bound encompasses the first three and can be proved using Dulmage and Mendelsohn's bound in the cyclicity classes.

We also generalize another bound by Kim [14], and a bound by Gregory–Kirkland–Pullman [12] which depend on Boolean rank.

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1 Wielandt's proof was published later in [17].

2 Denardo [9] later rediscovered their result.

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The six bounds mentioned above are stated in Theorems 2.11 and 2.14 after the requisite definitions. Our generalizations to weighted digraphs are stated in Main Theorems 1 and 2, and subsequently proved in Sections 3–7.

We exploit the natural connection between weighted digraphs and nonnegative matrices in the max (-times) algebra just as the bounds that we take for our starting points connect unweighted digraphs and Boolean matrices.

2. Preliminaries and statement of results

2.1. Digraphs, walks, and transients

Let us start with some definitions.

Definition 2.1 (*Digraphs*)**.** A *digraph* is a pair $\mathcal{G} = (N, E)$ where N is a set of nodes and $E \subseteq N \times N$ is a set of edges.

Definition 2.2 (*Walks and Cycles*)**.** A *walk* in $\mathcal{G}$ is a finite sequence $W = (i_0, i_1, \ldots, i_t)$ of nodes such that each pair (i_0, i_1), $(i_1, i_2), \ldots, (i_{t-1}, i_t)$ is an edge of $\mathcal{G}$ (that is, belongs to E). Here, the nodes i_0, resp. i_t are the *start* resp. the *end* nodes of the walk.

The number t is the *length* of the walk, and we denote it by $\ell(W)$.

When $i_0 = i_t$, the walk is *closed*. If, in a closed walk, none of the nodes except for the start and the end appear more than once, the walk is called a *cycle*. If no node appears more than once, then the walk is called a *path*. A walk is *empty* if its length is 0.

To a digraph $\mathcal{G} = (N, E)$ with $N = \{1, \ldots, n\}$, we can associate a Boolean matrix $A = (a_{i,j}) \in \mathbb{B}^{n\times n}$ defined by

$$a_{i,j} = \begin{cases} 0 & \text{if } (i,j) \notin E \\ 1 & \text{if } (i,j) \in E. \end{cases} \tag{2.1}$$

Conversely, one can associate a digraph to every square Boolean matrix. The connectivity in $\mathcal{G}$ is closely related to the Boolean matrix powers of A. By the *Boolean algebra* we mean the set $\mathbb{B} = \{0, 1\}$ equipped with the logical operations of conjunction $a \cdot b$ and disjunction $a \oplus b = \max(a, b)$, for $a, b \in \mathbb{B}$. The Boolean multiplication of two matrices $A \in \mathbb{B}^{m\times n}$ and $B \in \mathbb{B}^{n\times q}$ is defined by $(A \otimes B)_{i,j} := \bigoplus_{k=1}^{n}(a_{i,k} \cdot b_{k,j})$, and then we also have Boolean matrix powers $A^{\otimes t} := \underbrace{A \otimes \cdots \otimes A}_{t \text{ times}}$. The (i, j)th entry of $A^{\otimes t}$ is denoted by $a_{i,j}^{(t)}$.

The relation between Boolean powers of A and connectivity in $\mathcal{G}$ is based on the following fact: $a_{i,j}^{(t)} = 1$ if and only if $\mathcal{G}$ contains a walk of length t from i to j.

Let $\mathcal{G}$ be a digraph with associated matrix $A \in \mathbb{B}^{n\times n}$. The sequence of Boolean matrix powers $A^{\otimes t}$ is eventually periodic, that is, there exists a positive p such that

$$A^{\otimes(t+p)} = A^{\otimes t} \tag{2.2}$$

for all t large enough.

Definition 2.3 (*Eventual Period*)**.** Each p satisfying (2.2) is called an *eventual period* of A.

The set of nonnegative t satisfying (2.2) is the same for all eventual periods p.

Definition 2.4 (*Transient of Digraphs*)**.** The least nonnegative number t satisfying (2.2) for some (and hence for all) p is called the *transient (of periodicity) of* $\mathcal{G}$; we denote it by $T(\mathcal{G})$.

See [4] for general introduction to the theory of digraphs and [15] for a survey on their transients. In the literature, $T(\mathcal{G})$ is often called the *index of convergence*, or the *exponent* of $\mathcal{G}$.

Definition 2.5 (*Powers of Digraphs*)**.** The digraph associated with $A^{\otimes t}$ is denoted by $\mathcal{G}^t$. Such graphs will be further referred to as the *powers* of $\mathcal{G}$.

Definition 2.6 (*Cyclicity and Primitivity*)**.** For a strongly connected digraph $\mathcal{G}$, its *cyclicity* is the greatest common divisor of the lengths of all cycles of $\mathcal{G}$. If $d = 1$, then $\mathcal{G}$ is called *primitive*, otherwise it is called *imprimitive*.

The cyclicity d of $\mathcal{G}$ can be equivalently defined as the least eventual period p in (2.2). If $\mathcal{G}$ is strongly connected, then its cyclicity is the smallest eventual period of its sequence of powers. Let us recall the following basic observation from [4].

Theorem 2.7 ([4, Theorem 3.4.5])**.** *Let $\mathcal{G}$ be a strongly connected graph with cyclicity d. For each $k \geq 1$, graph $\mathcal{G}^k$ consists of* $\gcd(k, d)$ *isolated strongly connected components, and every component has cyclicity* $d/\gcd(k, d)$.

We have an important special case when $k = d$.