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Fractional acquisition in graphs

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ABSTRACT

Let G be a vertex-weighted graph in which each vertex has weight 1. Given a vertex u with positive weight and a neighbor v whose weight is at least the weight on u, a *fractional acquisition move* transfers some amount of weight at u from u to v. The *fractional acquisition number* of G, written $a_f(G)$, is the minimum number of vertices with positive weight after a sequence of fractional acquisition moves in G. In this paper, we determine the fractional acquisition number of all graphs: if G is an n-vertex path or cycle, then $a_f(G) = \lceil n/4 \rceil$; if G is connected with maximum degree at least 3, then $a_f(G) = 1$.

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1. Introduction

Consider an army that is deployed at a collection of bases, some of which are connected by roads. We wish to consolidate all of the troops at a single base. Troops are allowed to move only to neighboring bases with at least as many troops as their current base. Is it possible for all of the troops to reach a single base?

Let G be a vertex-weighted graph and let $w : V(G) \to \mathbb{R}_{\geq 0}$ be the weight assignment on G. In a weighted graph G, a *fractional acquisition move* transfers some positive amount of weight from a vertex u to a neighbor v, provided that the weight on v is at least the weight on u. The amount of weight transferred cannot exceed the weight on u. We refer to a succession of fractional acquisition moves as a *fractional protocol*; throughout this paper we assume that all protocols are finite in length. We study fractional protocols on graphs in which every vertex starts with weight 1. The *fractional acquisition number* of G, denoted as $a_f(G)$, is the minimum size of the set of vertices with positive weight after a fractional protocol on G. The weight assignment w is *feasible* if there is a fractional protocol that achieves w starting from the initial all-1s weight assignment.

Previous work on acquisition in graphs has focused on acquisition moves that transfer all of the weight from a vertex to its neighbor with at least the same weight; we call such an acquisition move a *total acquisition move*. Analogously, the minimum number of vertices in a graph G with positive weight after a sequence of total acquisition moves is the *total acquisition number* of G, denoted as $a_t(G)$. Lampert and Slater [1] proved that for $n \geq 2$, if G is a connected n-vertex graph, then $a_t(G) \leq (n+1)/3$. They also provided a lower bound on the total acquisition number of a connected graph depending on its degree sequence. LeSaulnier and West [3] characterized all graphs achieving equality for the total acquisition upper bound. Slater and Wang [4] proved that determining if the total acquisition number of a given graph G is 1 is an NP-complete problem and provided a linear time algorithm that determines the total acquisition number for caterpillars. LeSaulnier et al. [2] then provided a polynomial time algorithm to test $a_t(T) \leq k$ where T is a tree and k is any fixed positive integer. They also established numerous sufficient conditions for a graph to have total acquisition number 1.

In [1], Lampert and Slater used the term *consolidation* to refer to an acquisition move that moves an integer amount of weight from a vertex to its neighbor. Clearly, each total acquisition move is also a consolidation. A fractional acquisition

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move is a further generalization of a consolidation. Because each total acquisition move is also a fractional acquisition move, we have that $a_f(G) \leq a_t(G)$ for all G.

In this paper we determine the fractional acquisition number of all graphs. In Section 2, we prove that if G is a connected graph and $\Delta(G) \geq 3$, then $a_f(G) = 1$. This is a startling contrast to what is known about total acquisition numbers. All sufficient conditions in [2] for a graph to have total acquisition number 1 depend on dominating cliques or vertices of large degree whose neighborhoods are dominating sets. Such graphs have low diameter and strong structural requirements. More generally, it is conjectured in [2] that if G is an n-vertex graph with diameter 2, then $a_t(G)$ is bounded by an absolute constant (perhaps even 2), but the best known bound is that $a_t(G) \leq 32 \lg n \lg \lg n$. In contrast, adding a single pendant to a vertex of degree 2 in an $(n-1)$-vertex path yields an n-vertex graph with maximum degree 3, diameter $n-2$, and fractional acquisition number 1. We prove that trees with maximum degree at least 3 have fractional acquisition number 1 by inductively constructing a protocol that yields a weight distribution with certain desired properties. For arbitrary connected graphs with maximum degree at least 3 it is then sufficient to use the edge set of a spanning tree containing a vertex of degree at least 3.

In Section 3, we prove that the fractional acquisition numbers of the n-vertex path and the n-vertex cycle are $\lceil n/4 \rceil$. Combined with the result on connected graphs with maximum degree at least 3 this determines the fractional acquisition number of every connected graph, and consequently all graphs.

Interestingly, when G is a path or cycle, the fractional acquisition number and total acquisition number of G are equal (the total acquisition number is determined in [1]). Thus the freedom of fractional acquisition moves does not provide an advantage when the maximum degree of a graph is 2. The proof that $a_t(P_n) = a_t(C_n) = \lceil n/4 \rceil$ follows from the observations that each total acquisition move at most doubles the weight at a vertex and each edge can be used at most once by total acquisition moves (P_n and C_n denote the n-vertex path and cycle, respectively). Therefore the maximum amount of weight that a vertex in a path or cycle can acquire via total acquisition moves is 4, and the result follows. In contrast, in Section 3 we show that the maximum amount of weight that a vertex in P_n may acquire via fractional acquisition moves grows with n.

Though we determine the fractional acquisition number of all graphs, many interesting open questions remain, particularly with respect to the efficiency of fractional protocols. We present open questions and conjectures in Section 4.

For any undefined terminology, we refer the reader to [5].

2. General graphs

We determine the fractional acquisition number of graphs with maximum degree at least 3. If u and v are vertices in a tree T, then we let $T(u, v)$ denote the unique u, v-path in T. Also, we refer to the vertices in a tree T with degree at least 3 as *branch* vertices.

Theorem 1. *If G is a connected graph with $\Delta(G) \geq 3$, then $a_f(G) = 1$.*

In the proof of Theorem 1, we make extensive use of paths with weight assignments that allow all of the weight on the path to be acquired by a single vertex. An *ascending path* is a path $v_1 v_2 \ldots v_k$ with a weight assignment w such that $w(v_1) \leq w(v_2)$ and $w(v_i) < (v_{i+1})$ for $i \in \{2, \ldots, k-1\}$. When it is convenient, we will say that such a path P *ascends to* v_k, or that P is v_k*-ascending*. An ascending path P is *strictly ascending* if $w(v_1) < w(v_2)$. A weighted tree T is *ascending* if there is a vertex $v \in V(T)$ such that for every vertex u in the tree, $T(u, v)$ is v-ascending.

We will frequently use a protocol that moves weight along an ascending path. Let $P = v_1 \ldots v_k$ be a k-vertex path with a positive weight assignment w that ascends from v_1 to v_k. Let $c = \min(\{w(v_1)\} \cup \{w(v_{i+1}) - w(v_i) : 2 \leq i \leq k-1\})$. Define the *path protocol*, denoted by $\mathcal{A}(v_1, v_k)$, as follows. Transfer weight c from v_1 to v_k while moving no other weight; let step i in the protocol move weight c from v_i to v_{i+1}. After the ith step, the new weight on v_{i+1} is $w(v_{i+1}) + c$; since $w(v_{i+2}) \geq w(v_{i+1}) + c$, the protocol can continue. On step $k-1$, the "packet" of weight c reaches v_k and the protocol terminates. We denote ℓ repeated applications of the path protocol $\mathcal{A}(v_1, v_n)$ by $\mathcal{A}(v_1, v_n)^{\ell}$.

Lemma 2. *If a tree T has a feasible weight assignment w that ascends to a vertex r, then $a_f(T) = 1$ and r can acquire all of the weight in T.*

Proof. We use induction on the number of vertices in T with positive weight. If r is the only vertex with positive weight, then $a_f(T) = 1$. Otherwise, let u be a vertex with positive weight that is farthest from r. Use the path protocol to move weight to u from r; let c be the amount of weight transferred from u to r. Applying the protocol $\mathcal{A}(u, r)^{\lceil w(u)/c \rceil}$ leaves u with weight 0, r with weight $w(r) + w(u)$, and all other weights unchanged. Thus we have decreased the number of vertices with positive weight and we apply the induction hypothesis. □

To prove Theorem 1, we need only fractional acquisition moves that transfer rational amounts of weight; therefore we introduce a new model of fractional acquisition, which we call the *normalized* model. Let each vertex start with weight 0, and move finite positive amounts of weight, allowing negative weights on vertices. As with fractional acquisition moves, moving weight from u to v is valid only if the weight on v is at least the weight on u. A protocol of such moves is a *normalized protocol*; all normalized protocols we use are finite in length. In the normalized model, the vertex weights always sum to 0. A weight distribution where the weights sum to 0 is called a *normalized weight distribution*.

For the proof of Theorem 1, it suffices to consider only normalized acquisition moves that transfer integer amounts of weight. When convenient, we will refer to the units of weight that are moving around the graph as *chips*. When working

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