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The vertex leafage of chordal graphs

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ABSTRACT

Every chordal graph G can be represented as the intersection graph of a collection of subtrees of a host tree, a so-called *tree model* of G. This representation is not necessarily unique. The leafage $\ell(G)$ of a chordal graph G is the minimum number of leaves of the host tree of a tree model of G. The leafage is known to be polynomially computable.

In this contribution, we introduce and study the *vertex leafage*. The vertex leafage $v\ell(G)$ of a chordal graph G is the smallest number k such that there exists a tree model of G in which every subtree has at most k leaves. In particular, the case $v\ell(G) \leq 2$ coincides with the class of path graphs (vertex intersection graphs of paths in trees).

We prove for every fixed $k \geq 3$ that deciding whether the vertex leafage of a given chordal graph is at most k is NP-complete. In particular, we show that the problem is NP-complete on split graphs with vertex leafage of at most $k+1$. We further prove that it is NP-hard to find for a given split graph G (with vertex leafage at most three) a tree model with minimum total number leaves in all subtrees, or where maximum number of subtrees are paths. On the positive side, for chordal graphs of leafage at most ℓ, we show that the vertex leafage can be calculated in time $n^{O(\ell)}$.

Finally, we prove that every chordal graph G admits a tree model that realizes both the leafage and the vertex leafage of G. Notably, for every path graph G, there exists a path model with $\ell(G)$ leaves in the host tree and we describe an $O(n^3)$ time algorithm to compute such a path model.

1. Introduction

In the following text, a graph is always finite, simple, undirected, and loopless. A graph $G = (V, E)$ has vertex set $V(G)$ and edge set $E(G)$. We write uv for the edge $(u, v) \in E(G)$. We use $N_G(v)$ to denote the neighborhood of v in G, and write $N_G[v] = N_G(v) \cup \{v\}$. The degree of v in G is denoted by $\deg_G(v) = |N_G(v)|$. Where appropriate, we drop the index G, and write $N(v)$, $N[v]$, and $\deg(v)$, respectively. We use $G[X]$ to denote the subgraph of G induced by $X \subseteq V(G)$, and write $G - X$ for the graph $G[V(G) \setminus X]$. We use $G - v$ for $G - \{v\}$. We say that X is a *clique* of G if $G[X]$ is a complete graph, and X is an *independent set* of G if $G[X]$ has no edges.

A *tree model* of a graph $G = (V, E)$ is a pair $\mathscr{T} = (T, \{T_u\}_{u \in V})$ where T is a tree, called a *host tree*, each T_u is a *subtree* of T, and a pair uv is in E if and only if $V(T_u) \cap V(T_v) \neq \emptyset$. In other words, $\mathscr{T}$ consists of a host tree and a collection of its subtrees whose vertex intersection graph is G.

A graph is *chordal* if it does not contain an induced cycle of length four or more. It is well-known [1,7,21] that a graph is chordal if and only if it has a tree model. Any chordal graph admits possibly many different tree models.

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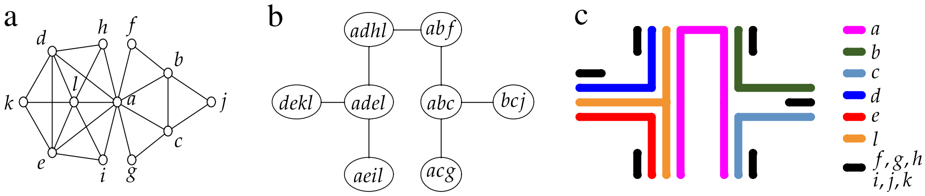

Fig. 1. (a) Example graph G with $\ell(G) = 4$ and $v\ell(G) = 3$, (b) example clique tree T of G, (c) tree model corresponding to (defined by) T.

For a tree T, let $\mathscr{L}(T)$ denote the set of its *leaves*, i.e., vertices of degree one. If T consists of a single node, we define $\mathscr{L}(T) = \emptyset$. In other words, we consider such a tree to have no leaves.

The *leafage* of a chordal graph G, denoted by $\ell(G)$, is defined as the smallest integer ℓ such that there exists a tree model of G whose host tree has ℓ leaves (see [15]). It is easy to see that $\ell(G) = 0$ if and only if G is a complete graph, and otherwise $\ell(G) \geq 2$. Moreover the case $\ell(G) \leq 2$ corresponds precisely to *interval graphs* (intersection graphs of intervals of the real line) [5]. In this sense, the leafage of a chordal graph G measures how close G is to being an interval graph.

In this paper, we introduce and study a similar parameter.

Definition 1. For a chordal graph $G = (V, E)$, the *vertex leafage* of G, denoted by $v\ell(G)$, is the smallest integer k such that there exists a tree model $\big(T, \{T_u\}_{u\in V}\big)$ of G where $|\mathscr{L}(T_u)| \leq k$ for all $u \in V$.

In other words, the vertex leafage of G seeks a tree model of G where each of the subtrees (corresponding to the vertices of G) has at most k leaves and the value of k is smallest possible. (See Fig. 1 for illustration.)

In the subsequent text, we shall say that a tree model of G *realizes the vertex leafage of* G to indicate that the tree model satisfies the conditions of Definition 1 for smallest possible k. Similarly, we shall say that a tree model of G *realizes the leafage of* G to indicate that the number of leaves in the host tree of the tree model is the smallest possible.

As in the case of leafage, the vertex leafage is a natural parameter related to some subclasses of chordal graphs previously studied in the literature. To see this, recall the class of vertex intersection graphs of paths in trees, also known as *path graphs* [8] (see also [2,14,16,18]). Now, observe that for a chordal graph G, we have $v\ell(G) = 0$ if G is a disjoint union of complete graphs, and otherwise $v\ell(G) \geq 2$. Moreover, $v\ell(G) \leq 2$ if and only if G is a path graph. Thus, the vertex leafage of a chordal graph G can be seen as a way to measure how close G is to being a path graph. Another connection comes from [11] where it is observed that in $O(kn)$ time one can find: an optimal coloring, a maximum independent set, a maximum clique, and an optimal clique cover of an n-vertex chordal graph G with vertex leafage k if a representation of G (a tree model realizing vertex leafage) is given.

In [8] it is shown that path graphs can be recognized in polynomial time. Currently, the best known recognition algorithms for path graphs run in $O(nm)$ time [2,18], where $n = |V(G)|$ and $m = |E(G)|$. In other words, for a chordal graph G, testing whether $v\ell(G) \leq 2$ can be performed in $O(nm)$ time.

Some other restrictions and variations on the standard tree model have also been studied. One such family of these variations is captured by the $[h, s, t]$-graphs (introduced in [13]) defined as follows: $G = (V, E)$ is an $[h, s, t]$-graph if there is a tree model $\big(T, \{T_u\}_{u\in V}\big)$ of G such that the maximum degree of T is at most h, the maximum degree of each of $\{T_u\}_{u\in V}$ is s, and uv is an edge of G if and only if T_u and T_v have at least t vertices in common. For more information on these graphs see [3,10].

We summarize the results of our paper in the following theorems.

Theorem 2. *For every $k \geq 3$, it is NP-complete to decide, for a split graph G whose vertex leafage is at most $k + 1$, if the vertex leafage of G is at most k.*

Theorem 3. *It is NP-complete to decide, for an integer p and a split graph G whose vertex leafage is at most 3,*

(i) *if there exists a tree model of G in which all but p subtrees are paths,*
(ii) *if there exists a tree model of G where the total number of leaves in all subtrees is at most p.*

Theorem 4. *For every $\ell \geq 2$, there exists an $n^{O(\ell)}$ time algorithm that, given an n-vertex chordal graph G with $\ell(G) \leq \ell$, computes the vertex leafage of G and constructs a tree model of G that realizes the vertex leafage of G.*

Theorem 5. *There exists an $O(n^3)$ time algorithm that, given an n-vertex chordal graph $G = (V, E)$ and a tree model $(T, \{T_u\}_{u\in V})$ of G, computes a tree model $(T^*, \{T^*_u\}_{u\in V})$ of G such that*

(i) $|\mathscr{L}(T^*_u)| \leq |\mathscr{L}(T_u)|$ *for all* $u \in V$,
(ii) $|\mathscr{L}(T^*)| = \ell(G)$.

Corollary 6. *For every chordal graph $G = (V, E)$, there exists a tree model $(T^*, \{T^*_u\}_{u\in V})$ such that*

(i) $|\mathscr{L}(T^*_u)| \leq v\ell(G)$ *for all* $u \in V$.
(ii) $|\mathscr{L}(T^*)| = \ell(G)$.

In other words, such a tree model realizes both the leafage and the vertex leafage of G.