Discrete Applied Mathematics 168 (2014) 26–33

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On opposition graphs, coalition graphs, and bipartite permutation graphs

Van Bang Le

Universität Rostock, Institut für Informatik, Rostock, Germany

ARTICLE INFO

Article history:
Received 24 September 2012
Received in revised form 16 November 2012
Accepted 23 November 2012
Available online 20 December 2012

Keywords:
Perfectly orderable graph
One-in-one-out graph
Bipartite permutation graph
Opposition graph
Coalition graph

ABSTRACT

A graph is an opposition graph, respectively, a coalition graph, if it admits an acyclic orientation which puts the two end-edges of every chordless 4-vertex path in opposition, respectively, in the same direction. Opposition and coalition graphs have been introduced and investigated in connection to perfect graphs. Recognizing and characterizing opposition and coalition graphs still remain long-standing open problems. The present paper gives characterizations for co-bipartite opposition graphs and co-bipartite coalition graphs, and for bipartite opposition graphs. Implicit in our argument is a linear time recognition algorithm for these graphs. As an interesting by-product, we find new submatrix characterizations for the well-studied bipartite permutation graphs.

1. Introduction and preliminaries

Chvátal [5] proposed to call a linear order $<$ on the vertex set of an undirected graph G *perfect* if the greedy coloring algorithm applied to each induced subgraph H of G gives an optimal coloring of H: consider the vertices of H sequentially by following the order $<$ and assign to each vertex v the smallest color not used on any neighbor u of v, $u < v$. A graph is *perfectly orderable* if it admits a perfect order. Chvátal proved that $<$ is a perfect order if and only if there is no chordless path with four vertices a, b, c, d and three edges ab, bc, cd (written P_4 $abcd$) with $a < b$ and $d < c$. He also proved that perfectly orderable graphs are perfect.[1] The class of perfectly orderable graphs properly contains many important, classical classes of perfect graphs such as chordal graphs and comparability graphs. Perfectly orderable graphs have been extensively studied in the literature; see Hoàng's comprehensive survey [12] for more information.

In [6], Chvátal pointed out a somewhat surprising connection between perfectly orderable graphs and a well-known theorem in mathematical programming. A matrix M is a *submatrix* of a matrix N if M can be obtained from N by deleting some columns and rows in N; N is M*-free* if there exist permutations of the rows and columns of N such that the permuted matrix (which we will again denote by N) does not contain M as a submatrix. A 0/1 matrix is *totally balanced* if it does not contain, as a submatrix, the edge-vertex incidence matrix of a cycle of length at least three. The following characterization of totally balanced 0/1 matrices is well known; write $\Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Theorem 1 ([1,14,16]). *A 0/1 matrix is totally balanced if and only if it is Γ-free.*

The *bimatrix* $M_G = (m_{ij})$ of a bipartite graph $G = (X, Y, E)$ is a 0/1 matrix whose rows represent the vertices in the color class X and whose columns represent the vertices in the color class Y in such a way that $m_{ij} = 1$ if and only if the vertex represented by the ith row is adjacent to the vertex represented by the jth column. It can be verified (cf. also the proof of

E-mail address: le@informatik.uni-rostock.de.

[1] A graph is *perfect* if the chromatic number and the clique number are equal in every induced subgraph.

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doi:10.1016/j.dam.2012.11.020

Theorem 5) that the complement $\overline{G}$ of G is perfectly orderable if and only if M_G is Γ-free, and that G is *chordal bipartite* (that is, G does not contain any chordless cycle with at least six vertices) if and only if M_G is totally balanced. Thus, Theorem 1 can be reformulated as follows:

Theorem 2 ([1,6,14,16])**.** *For a bipartite graph $G = (X, Y, E)$, the following statements are equivalent:*

(i) *$\overline{G}$ is perfectly orderable.*
(ii) *G is chordal bipartite.*
(iii) *M_G is Γ-free.*

In particular, recognizing if a co-bipartite graph (the complement of a bipartite graph) is perfectly orderable reduces to recognizing if a graph is chordal bipartite, which can be done in quadratic time (cf. [22]).

In general, recognizing perfectly orderable graphs is NP-complete [18] (see also [11]). Also, no characterization of perfectly orderable graphs by forbidden induced subgraphs is known. These facts have motivated researchers to study subclasses of perfectly orderable graphs; see, e.g., [8,12,13] and the literature given there. Observe that a linear order $<$ corresponds to an acyclic orientation by directing the edge xy from x to y if and only if $x < y$. Thus, a graph is perfectly orderable if and only if it admits an acyclic orientation such that no chordless path P_4 is oriented of type 0 depicted in Fig. 1; equivalently, every P_4 is oriented of type 1, 2, or 3.

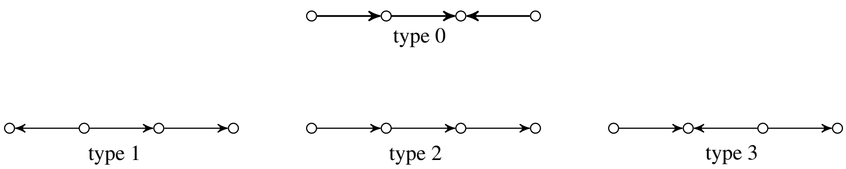

Fig. 1. Four types of oriented P_4.

One of the natural subclass of perfectly orderable graphs for which the recognition complexity, as well as an induced subgraph characterization are still unknown is the following (cf. [12,13]).

Definition 1. A graph is a *coalition graph* if it admits an acyclic orientation such that every induced P_4 *abcd* has the end-edges *ab* and *cd* oriented in the 'same way', that is, every oriented P_4 is of type 2 or 3.

Equivalently, a graph is a coalition graph if it admits an order $<$ on its vertex set such that every induced P_4 *abcd* has $a < b$ if and only if $c < d$. In [12], coalition graphs are called one-in-one-out graphs. Examples of coalition graphs include comparability graphs, hence all bipartite graphs.

A related graph class has been introduced by Olariu in [19]:

Definition 2. A graph is an *opposition graph* if it admits an acyclic orientation such that every induced P_4 *abcd* has the end-edges *ab* and *cd* oriented 'in opposition', that is, every oriented P_4 is of type 0 or 1.

Equivalently, a graph is an opposition graph if it admits an order $<$ on its vertex set such that every P_4 *abcd* has $a < b$ if and only if $d < c$. Olariu [19] proved that opposition graphs are perfect. He also conjectures [20] that not all opposition graphs are perfectly orderable. Examples of opposition graphs include all split graphs. The recognition and characterization problems for opposition graphs are still open. A natural subclass of opposition graphs consists of those admitting an acyclic orientation in which every P_4 is oriented as type 1 (equivalently, every P_4 is oriented as type 0) has been characterized by forbidden induced subgraphs in [10,13], and has been recognized in $O(n^{3.376})$ time in [8]; n is the vertex number of the input graph.

The purpose of the present paper is to find characterizations for co-bipartite coalition graphs and co-bipartite opposition graphs, similar to those stated in Theorem 2, and for bipartite opposition graphs. In doing so, we will find new characterizations for the well-studied bipartite permutation graphs.

Definition 3. A graph is a *permutation graph* if there is some pair π_1, π_2 of permutations of the vertex set such that there is an edge xy if and only if x precedes y in one permutation in $\{\pi_1, \pi_2\}$ and y precedes x in the other permutation. A *bipartite permutation graph* is a permutation graph that is also bipartite.

Bipartite permutation graphs admit several characterizations and can be recognized in linear time [21]; see [22] for more information on permutation and bipartite permutation graphs. The following characterizations of bipartite permutation graphs follow first from the fact that permutation graphs are exactly the comparability graphs which are also co-comparability graphs, and second, from Gallai's subgraph characterization of comparability graphs [9,17] that C_{2k}, $k \geq 3$, and B_1, B_2 and B_3 depicted in Fig. 2 are the only minimal bipartite graphs which are not co-comparability graphs.

Theorem 3 (Folklore)**.** *Let G be a bipartite graph. Then G is a permutation graph if and only if $\overline{G}$ is a comparability graph if and only if G is a $\{B_1, B_2, B_3\}$-free chordal bipartite graph.*