Discrete Applied Mathematics 168 (2014) 34–39

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Square roots of minor closed graph classes

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ARTICLE INFO

Article history:
Received 29 February 2012
Received in revised form 23 May 2013
Accepted 27 May 2013
Available online 22 June 2013

Keywords:
Square roots of graphs
Branch-width
Carving-width
Graph minors

ABSTRACT

Let $\mathcal{G}$ be a graph class. The *square root* of $\mathcal{G}$ contains all graphs whose squares belong in $\mathcal{G}$. We prove that if $\mathcal{G}$ is non-trivial and minor closed, then all graphs in its square root have carving-width bounded by some constant depending only on $\mathcal{G}$. As a consequence, every square root of such a graph class has a linear time recognition algorithm.

1. Introduction

Let $\mathcal{G}$ be a graph class. The *square root* of $\mathcal{G}$ is defined as the graph class

$$\sqrt{\mathcal{G}} = \{G \mid G^2 \in \mathcal{G}\},$$

where the *square* G^2 of a graph G is the graph obtained from G after adding edges between all pairs of vertices that share a common neighbor.

In [3], Harary, Karp, and Tutte provided a complete characterization of the graphs in $\sqrt{\mathcal{P}}$ where $\mathcal{P}$ is the class of all planar graphs. Notice that planar graphs are *minor closed*, i.e. a minor of every graph in $\mathcal{P}$ also belongs in $\mathcal{P}$. Minor closeness is a very general property that is satisfied by a great variety of graph classes; see e.g. [6].

According to the characterization of [3], all graphs in $\sqrt{\mathcal{P}}$ are outerplanar and have bounded degree. This implies that graphs in $\sqrt{\mathcal{P}}$ have a very specific "tree-like" structure and it is a natural question whether this is the case for more general minor closed graph classes. In this paper we extend this result, in the sense that the same tree-like property holds for every minor closed graph class that is non-trivial (i.e. that does not contain all graphs). In fact, we prove (in Section 3) that, in this case, the correct "tree-likeness" property is given by the parameter of carving-width, introduced by Seymour and Thomas in [11]. As a consequence, we prove in Section 4 that the square root of any non-trivial minor closed graph class has a linear time recognition algorithm. This extends the algorithmic results of [5] where a linear time algorithm was given for recognizing the square roots of planar graphs.

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1 The second author was Co-financed by the European Union (European Social Fund — ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: "Thales. Investing in knowledge society through the European Social Fund".

0166-218X/$ – see front matter
http://dx.doi.org/10.1016/j.dam.2013.05.026

2. Definitions

We next give some definitions that are necessary in order to formally define carving width. This will permit us to give the formal statement of our combinatorial result.

Boundaries in graphs and hypergraphs. In this paper we deal with graphs and hypergraphs. For a (hyper)graph G we denote by $V(G)$ its vertex set and by $E(G)$ the set of its (hyper)edges. If $S \subseteq V(G)$ (resp. $F \subseteq E(G)$) we denote $\overline{S} = V(G) \setminus S$ (resp. $\overline{F} = E(G) \setminus F$).

Given a vertex set $S \subseteq V(G)$, let $E_G(S)$ be the set of hyperedges containing vertices in S. For simplicity we also denote $E_G(v) = E_G(\{v\})$. We also define $\Delta(G) = \max\{|E_G(v)| \mid v \in V(G)\}$. Given a set $S \subseteq V(G)$, we define

$$\partial_G(S) = E_G(S) \cap E_G(\overline{S}).$$

Notice that ∂_G is a symmetric function, i.e. for every $S \subseteq V(G)$, it holds that $\partial_G(S) = \partial_G(\overline{S})$. Also given a set $F \subseteq E(G)$, we set

$$\partial^*_G(F) = \left(\bigcup_{f \in F} f\right) \cap \left(\bigcup_{f \in \overline{F}} f\right).$$

Given a hypergraph G we define its *dual* as the hypergraph

$$G^* = (E(G), \{E_G(v) \mid v \in V(G)\}).$$

Notice that the hypergraphs G and G^* have the same incidence graph with the roles of their two parts reversed. Given a set $S \subseteq V(G)$ (resp. $F \subseteq E(G)$) we denote by S^* (resp. F^*) their dual hyperedges (resp. vertices) in G^*.

Using duality, we also define $\Delta^*(G) = \Delta(G^*)$. Clearly, for a simple graph G, $\Delta^*(G) = 2$. Moreover, the above definitions imply that for every $F \subseteq E(G)$, $(\partial^*_G(F))^* = \partial_{G^*}(F^*)$.

A graph H is a *minor* of a graph G, and we write $H \leq G$, if H can be obtained for some subgraph of G after contracting edges (the contracting an edge $e = \{x, y\}$ is the operation that removes x and y from G and introduces a new vertex v_e that is made adjacent with all the neighbors of x and y in G, except from x and y). A graph class $\mathcal{G}$ is *minor closed* if every minor of a graph in $\mathcal{G}$ is also a graph in $\mathcal{G}$.

Carving-width. Given a tree T we denote the set of its leaves by $L(T)$ and we call it *ternary* if all vertices in $V(T) \setminus L(T)$ have degree 3. A *carving decomposition* of a hypergraph G is a pair (T, ρ), where T is a ternary tree and ρ is a bijection from $V(G)$ to $L(T)$. The *bridge function* $\beta : E(T) \to 2^{E(G)}$ of a carving decomposition maps every edge e of T to the set $\partial_G(\rho^{-1}(L(T')))$ where T' is one of the two connected components of $T \setminus e$. The *width* of (T, ρ) is equal to $\max_{e \in E(T)} |\beta(e)|$ and the *carving-width* of G, $\mathbf{cw}(G)$, is the minimum width over all carving decompositions of G. The following observation is a direct consequence of the definitions.

Observation 1. *For every hypergraph G, it holds that $\Delta(G) \leq \mathbf{cw}(G)$.*

The main combinatorial result of this paper is the following.

Theorem 1. *For every non-trivial minor closed graph class $\mathcal{G}$ there is a constant $c_{\mathcal{G}}$ such that all graphs in $\sqrt{\mathcal{G}}$ have carving-width at most $c_{\mathcal{G}}$.*

The proof of Theorem 1 uses the parameter of branch-width defined in [8].

Branch-width. A *branch decomposition* of a graph G is a pair (T, τ), where T is a ternary tree and τ is a bijection from $E(G)$ to $L(T)$. The *boundary function* $\omega : E(T) \to 2^{V(G)}$ of a branch decomposition maps every edge e of T to the set $\partial^*_G(\rho^{-1}(L(T')))$ where T' is one of the two connected components of $T - \{e\}$. The *width* of (T, τ) is equal to $\max_{e \in E(T)} |\omega(e)|$ and the *branch-width* of G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G.

The following observation is a direct consequence of the duality between the functions ∂_G and ∂^*_G.

Observation 2. *For every hypergraph G it holds that $\mathbf{bw}(G) = \mathbf{cw}(G^*)$.*

3. Walls and squares

Walls. A *wall of height k*, $k \geq 1$, is obtained from a $((k+1) \times (2k+2))$-grid with vertices (x, y), $x \in \{0, \ldots, 2k+1\}$, $y \in \{0, \ldots, k\}$, after removing the "vertical" edges $\{(x, y), (x, y+1)\}$ for odd $x + y$. We denote such a wall by W_k. A *subdivided wall of height k* is obtained by the wall W_k with some edges of W_k replaced by paths without common internal vertices. If, in such a subdivided wall, all edges have been subdivided at least once, then we say that it is *properly subdivided*. We also say that a graph H is *topological minor* of a graph G if some subdivision of H is a subgraph of G.

The following result follows from the results in [7,8].

Proposition 1 ([7,8]). *There is a function $g : \mathbb{N} \to \mathbb{N}$ such that every graph G with branchwidth at least $g(k)$ contains the $(k \times k)$-grid as a minor.*