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Digraphs of bounded elimination width☆

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ABSTRACT

Hunter and Kreutzer recently introduced the digraph width measure Kelly-width. The measure is based on a vertex elimination process that aims at bounding the out-degree of a vertex during the elimination. We refine the Kelly-width measure by bounding both the out-degree and the in-degree of a vertex. We show that the elimination process and a subgraph characterisation of Kelly-width naturally generalise to the refined notion. The main result of the paper is a game characterisation of the refined measure, which is a surprisingly non-trivial generalisation of the game for Kelly-width.

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1. Introduction

In algorithmic graph theory, width measures are used to describe the complexity of problems. Typical results are of the following form: graphs of bounded width admit an efficient solution. A very successful such width measure, i.e., a width measure with many applications and positive results, is the treewidth for undirected graphs [5]. Other width measures for undirected graphs of similar algorithmic properties are clique-width and rank-width [6,16]. An objective in algorithmic digraph theory is to define width measures that admit analogous results: digraphs of small width admit efficient solutions. Several digraph width measures have been proposed during recent years, most notably the directed tree-width [11], DAG-width [2,15,3], and Kelly-width [10], and they admit characterisations through cops-and-robber games [11,2,15,3,10]. Cops-and-robber games provide a unified approach to width measures. They define a graph separation notion and can be used as the basis in the design of dynamic-programming algorithms. It seemed that width measures with cops-and-robber game characterisations would eventually lead to treewidth analogues. This hope, however, has recently been deadened by the results of Ganian et al. [8], who showed that, in simple terms, a strong digraph width measure with a cops-and-robber game characterisation must be similar to the treewidth measure.

Despite the negative result by Ganian et al. from [8] about the existence of good digraph width measures that are different from the treewidth, we continue the study of digraph width measures that admit a cops-and-robber game characterisation. We introduce a measure that is closely related to the Kelly-width, but, instead of considering and bounding only the out-degree of vertices, our measure bounds both the out-degree and the in-degree of a vertex. We show that several characterisations of Kelly-width extend to characterisations of the new width measure. The subgraph characterisation and the characterisation through a vertex elimination process are direct and natural generalisations. These and further easy results are presented in Section 2.

The main result of this paper is a cops-and-robber game characterisation of our digraph width measure. From the two characterisation results of Section 2, it would be natural to expect that also the cops-and-robber game for Kelly-width carries

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over to our width measure in a straightforward way. This, however, is not the case. Briefly, the Kelly-width can be defined as the smallest number of cops that catch an inert and invisible robber [10]. The robber only moves along directed paths in the direction of the arcs, and thus, cops need to block such directed paths. For our width measure, the robber is also bound to escape along directed paths, but may ignore the direction of the path. Thus, cops need to block directed paths against a robber that moves "forward" or "backward". This defines two types of cop. It turns out that these two types of cop are not enough to capture our width measure. A third type of cop is needed, one which has very special powers and restrictions. The game is presented in Section 3. In Sections 3 and 4, we prove the characterisation result. Most of our proofs are based on ideas from the analogous proofs by Hunter and Kreutzer [10].

Our study of digraph width measures is led by two objectives. Acyclic digraphs have bounded DAG-width and Kelly-width. Many graph problems are hard on acyclic digraphs, for example, all hard problems on undirected graphs, which can be defined as digraph problems and ignore the actual directions of the arcs involved. With our width measure, an acyclic digraph can have arbitrarily large width. This finer-grained measure may lead to algorithmic results of the desired form. A second objective focuses on more theoretical purposes. As has already been mentioned, the cops-and-robber game defines a digraph separation notion. As for most undirected graph notions, a widely accepted and useful separation notion for digraphs is not known. Cops-and-robber games may help in defining a useful such separation notion.

2. Digraphs of bounded elimination width

We consider simple and finite graphs. Mostly, we consider directed graphs. In a few cases, for relating our results to other results, we will also consider undirected graphs. The necessary definitions for undirected graphs are standard, can be found in textbooks on graphs and are analogous to the ones given in the following for directed graphs.

We assume that our graphs are not empty. A directed graph is called a *digraph* for short. Let $G = (V, A)$ be a digraph. The *vertex set* of G is $V(G) = V$ and the *arc set* of G is $A(G) = A$. Arcs of G are denoted as (u, v), where u is an *in-neighbour* of v in G and v is an *out-neighbour* of u in G. Our digraphs have non-empty vertex sets, and they have no loops, i.e., they have no arcs of the form (u, u). The *in-neighbourhood* of a vertex u of G, denoted as $N_G^{\text{in}}(u)$, is the set of the in-neighbours of u in G, and the *out-neighbourhood* of u, denoted as $N_G^{\text{out}}(u)$, is the set of the out-neighbours of u in G. A digraph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $A(H) \subseteq A(G)$. If H is a subgraph of G and $V(H) = V(G)$, then H is a *partial graph* of G. For a set X of vertices of G, $G \setminus X$ is the subgraph G' of G on vertex set $V(G) \setminus X$, and for every ordered vertex pair u, v of G', (u, v) is an arc of G' if and only if (u, v) is an arc of G. An *induced subgraph* of G is a digraph $G \setminus X$ for $X \subseteq V(G)$. For $Y \subseteq V(G)$, $G[Y]$ denotes the induced subgraph $G \setminus (V(G) \setminus Y)$. For a vertex x of G, $G - x$ denotes the digraph $G \setminus \{x\}$. A set Y of arcs is *suitable* for G if for every arc (u, v) in Y, u and v are vertices of G. For a suitable set Y of arcs for G, $G \cup Y$ is the digraph on vertex set $V(G)$ and with arc set $A(G) \cup Y$. Note that G is a partial graph of $G \cup Y$. For u and v two vertices of G, a *directed u, v-path of G* is a sequence $(x_0, \ldots, x_r)$ of pairwise different vertices of G such that $x_0 = u$ and $x_r = v$ and $(x_{i-1}, x_i) \in A(G)$ for every index i with $1 \leq i \leq r$.

Let G be a digraph. A *vertex layout* for G is a linear ordering $\beta = \langle x_1, \ldots, x_n \rangle$ of the vertices of G. Note that there is a 1-to-1 correspondence between the vertices of G and the indices $1, \ldots, n$, specified by the vertex layout β. For an ordered vertex pair u, v of G, we write $u \preccurlyeq_\beta v$ if there are indices i, j with $1 \leq i \leq j \leq n$ and $u = x_i$ and $v = x_j$. If $u \preccurlyeq_\beta v$ and $i \neq j$, i.e., if $u \preccurlyeq_\beta v$ and $u \neq v$, then we write $u \prec_\beta v$.

Let G be a digraph. Let A and B be not necessarily disjoint sets of vertices of G. We call (A, B) a *d-clique* of G if, for every ordered vertex pair u, v of G with $u \in A$ and $v \in B$ and $u \neq v$, (u, v) is an arc of G. Note that A or B may also be empty sets. The d-clique notion previously appeared, implicitly or explicitly, in [9,10,13]. Based on the d-clique notion, we define a parameterised class of digraphs. The definition summarises and refines previous definitions by Haskins and Rose [9] and by Hunter and Kreutzer [10].

Definition 2.1. Let a and b be integers with $a \geq 0$ and $b \geq 0$. Let G be a digraph, and let $\beta = \langle x_1, \ldots, x_n \rangle$ be a vertex layout for G. Then, G is a *perfect elimination digraph of width at most (a, b) with construction sequence β* if $n = 1$ or if $n \geq 2$ and the following three conditions are satisfied:

(1) $G - x_1$ is a perfect elimination digraph of width at most (a, b) with construction sequence $\langle x_2, \ldots, x_n \rangle$,
(2) $(N_G^{\text{in}}(x_1), N_G^{\text{out}}(x_1))$ is a d-clique of G,
(3) $|N_G^{\text{in}}(x_1)| \leq a$ and $|N_G^{\text{out}}(x_1)| \leq b$.

For a and b integers with $a \geq 0$ and $b \geq 0$ and G a digraph, G is a *perfect elimination digraph of width at most (a, b)* if G has a vertex layout β such that G is a perfect elimination digraph of width at most (a, b) with construction sequence β. We sometimes refer to perfect elimination digraphs of width at most (a, b) by "perfect elimination digraphs of bounded width", if the actual bound (a, b) on the width is not important. A digraph G is a *perfect elimination digraph* if there are integers a, b with $a, b \geq 0$ such that G is a perfect elimination digraph of width at most (a, b). Perfect elimination digraphs were introduced by Haskins and Rose as a directed analogue of perfect elimination undirected graphs [9].

Our first result about perfect elimination digraphs of bounded width shows that these digraphs exist for every width bound, that classes of perfect elimination digraphs of different width do not coincide, and that the width defines an infinite hierarchy on the perfect elimination digraphs. Let G be a digraph. The *reverse digraph* of G, $\text{rev}(G)$, is the digraph on vertex set $V(G)$, and, for every ordered vertex pair u, v of G, (u, v) is an arc of $\text{rev}(G)$ if and only if (v, u) is an arc of G.

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