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# On the tree-depth of random graphs☆

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ABSTRACT

Tree-depth is a parameter introduced under several names as a measure of sparsity of a graph. We compute asymptotic values of the tree-depth of a random graph on $n$ vertices where each edge appears independently with probability $p$. For dense graphs, $np \to +\infty$, the tree-depth of a random graph $G$ is *aas* $\mathrm{td}(G) = n - O(\sqrt{n/p})$. Random graphs with $p = c/n$, have *aas*linear tree-depth when $c > 1$, the tree-depth is $\Theta(\log n)$ when $c = 1$ and $\Theta(\log \log n)$ for $c < 1$. The result for $c > 1$ is derived from the computation of tree-width and provides a more direct proof of a conjecture by Gao on the linearity of tree-width recently proved by Lee, Lee and Oum (2012) [15]. We also show that, for $c = 1$, every width parameter is *aas*constant, and that random regular graphs have linear tree-depth.



## 1. Introduction

An elimination tree of a connected graph $G$ is a rooted tree on the set of vertices such that there are no edges in $G$ between vertices in different branches of the tree. The natural elimination scheme provided by this tree is used in many graph algorithmic problems where two non adjacent subsets of vertices can be managed independently. One good example is the Cholesky decomposition of symmetric matrices (see [24,7,16,22]). Given an elimination tree, a distributed algorithm can be designed which takes care of disjoint subsets of vertices in different parallel processors. Starting from the furthest level from the root, it proceeds by exposing at each step the vertices at a given depth. Then the algorithm runs using the information coming from its children subtrees, which have been computed in previous steps. Observe that the vertices treated in different processors are independent and thus, there is no need to share information among them. Depending on the graph, this distributed algorithm can be more efficient than the sequential one. In fact, its complexity is given by the height of the elimination tree used by the algorithm. This motivates the study of the minimum height of an elimination tree of $G$. This natural parameter has been introduced under numerous names in the literature: rank function [21], vertex ranking number (or ordered coloring) [6], weak coloring number [13], but its study was systematically undertaken by Nešetřil and Ossona de Mendez under the name of *tree-depth*.

The tree-depth $\mathrm{td}(G)$ of a graph $G$ is a measure introduced by Nešetřil and Ossona de Mendez [19] in the context of bounded expansion classes. The notion of tree-depth is closely connected to tree-width. The tree-width of a graph can be seen as a measure of closeness to a tree, while the tree-depth takes also into account the diameter of the tree (see Section 2 for a precise definition and [20] for an extensive account of the meaning and applications of this parameter).

Bounded expansion classes are defined in terms of shallow minors and its connection to tree-depth is highlighted by the following result. The $k$-th chromatic number of a graph $\chi_k(G)$ is defined as the minimum number of colors needed to color a graph in such a way that each subgraph $H$ induced by any set of $i$ classes of colors, $i \leq k$, satisfy $\mathrm{td}(H) \leq i$. When we look

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at a bounded number of chromatic classes we see a simple structure. Observe that $\chi_1(G)$ is the ordinary chromatic number and $\chi_2(G)$ is the so-called star chromatic number [1]. The main theorem in this context states that a class of graphs $\mathcal{C}$ has bounded expansion if and only if $\limsup_{G\in\mathcal{C}} \chi_k(G) < +\infty$ for any fixed $k > 0$. This gives another motivation to study the tree-depth.

Randomly generated graphs have been widely used as benchmarks for testing distributed algorithms and therefore it is useful to characterize the elimination tree of such graphs. The main goal of this paper is to give asymptotically tight values for the minimum height of an elimination tree of a random graph.

A *random graph* $G \sim \mathcal{G}(n, p)$ has $n$ vertices and every pair of vertices is chosen independently to be an edge with probability $p$.

For any graph property $\mathcal{P}$, we say that $\mathcal{P}$ holds *asymptotically almost sure* (*aas*) in $G \sim \mathcal{G}(n, p)$, if

$$\lim_{n\to+\infty} \Pr(G \text{ satisfies } \mathcal{P}) = 1.$$

Throughout the paper, all the results and statements concerning random graphs must be understood in the asymptotically almost sure sense. We will occasionally make use of the $\mathcal{G}(n, m)$ model of random graphs, where a labeled graph with $n$ vertices and $m$ edges is chosen with the uniform distribution. As it is well-known, the two models are closely connected and most of the statements can be transferred from one model to the other one.

Our first result gives the value of tree-depth for dense random graphs.

**Theorem 1.1.** *Let $G \sim \mathcal{G}(n, p)$ be a random graph with $np \to +\infty$, then $G$ satisfies aas*

$$\mathrm{td}(G) = n - O\left(\sqrt{\frac{n}{p}}\right).$$

Theorem 1.1 implies that random graphs with a superlinear number of edges have a tree structure similar to the one of the complete graph. Actually our proof of Theorem 1.1 provides the same result for tree-width. To our knowledge, the tree-width of a dense random graph has not been studied before.

Nešetřil and Ossona de Mendez showed that a sparse random graph $G(n, c/n)$ belongs *aas* to a bounded expansion class (see [20, Theorem 13.4]). Our main result is the computation of the tree-depth of sparse random graphs.

**Theorem 1.2.** *Let $G \sim \mathcal{G}(n, p)$ be a random graph with $p = \frac{c}{n}$, with $c > 0$,*

(1) *if $c < 1$, then aas* $\mathrm{td}(G) = \Theta(\log\log n)$
(2) *if $c = 1$, then aas* $\mathrm{td}(G) = \Theta(\log n)$
(3) *if $c > 1$, then aas* $\mathrm{td}(G) = \Theta(n)$.

Henceforth we denote the logarithm in base two by log while we will use ln for the natural logarithm.

The last part of this theorem is closely related to a conjecture of Gao announced in [11] on the linear behavior of tree-width for random graphs with $c = 2$, inspired by the results of Kloks in [14]. This conjecture has been recently proved by Lee et al. [15]. They show that the tree-width is linear for any $c > 1$ as a corollary of their result on the rank-width of random graphs. Here we give a proof of Theorem 1.2.3 which also provides a proof of Gao's conjecture, giving an explicit lower bound on the tree-width. Our proof uses, as the one in [15], the same deep result of Benjamini et al. [2] on the existence of a linear order expander in a sparse random graph for $c > 1$.

The paper is organized as follows. In Section 2, we define the notion of tree-depth and give some useful results concerning this parameter. Section 3 contains the proof of Theorem 1.1, which uses the relation connecting tree-width with balanced partitions. Finally Theorem 1.2 will be proved in Section 4. For $c < 1$ the result follows from the fact that the random graph is a collection of trees and unicyclic graphs of logarithmic order, which gives the upper bound, and that there is one of these components with large diameter with respect to its order, providing the lower bound. For $c = 1$ we show that the giant component in the random graph has just a constant number of additional edges exceeding the order of a tree, which gives the upper bound, and rely on a result of Nachmias and Peres [18] on the concentration of the diameter of the giant component to obtain the lower bound. Finally, as we have already mentioned, for $c > 1$ the result follows readily from the existence of an expander of linear order in a sparse random graph for $c > 1$, a fact proved in Benjamini et al. [2].

## 2. Tree-depth

Let $T$ be a rooted tree. The *height* of $T$ is defined as the number of vertices of the longest rooted path. The *closure* of $T$ is the graph that has the same set of vertices and an edge between every pair of vertices such that one is an ancestor of the other in the rooted tree.

The tree $T$ is an *elimination tree* of a connected graph $G$ if $G$ is a subgraph of its closure. The *tree-depth* of a connected graph $G$ is defined to be the minimum height of an elimination tree of $G$. Some examples are shown in Fig. 1.

This definition can be extended to non-connected graphs. Suppose that $G$ has connected components $H_1, \ldots, H_s$. Then,

$$\mathrm{td}(G) = \max_{1\le i\le s} \mathrm{td}(H_i). \tag{1}$$