Discrete Applied Mathematics 166 (2014) 1–13

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Reputation games for undirected graphs

David Avis [a,b,*], Kazuo Iwama [a], Daichi Paku [a]

[a] *School of Informatics, Kyoto University, Japan*
[b] *School of Computer Science, McGill University, H3A 2A7, Canada*

ARTICLE INFO

Article history:
Received 25 May 2012
Received in revised form 2 August 2013
Accepted 27 September 2013
Available online 7 November 2013

Keywords:
PageRank
Game theory
Nash equilibria
Fractional optimization

ABSTRACT

J. Hopcroft and D. Sheldon originally introduced network reputation games to investigate the self-interested behavior of web authors who want to maximize their PageRank on a *directed* web graph by choosing their outlinks in a game theoretic manner. They give best response strategies for each player and characterize properties of web graphs which are Nash equilibria. In this paper we consider three different models for PageRank games on *undirected* graphs such as certain social networks. In undirected graphs players may delete links at will, but typically cannot add links without the other player's permission. In the *deletion-model* players are free to delete any of their bidirectional links but may not add links. We study the problem of determining whether the given graph represents a Nash equilibrium or not in this model. We give an $O(n^2)$ time algorithm for a tree, and a parametric $O(2^k n^4)$ time algorithm for general graphs, where k is the maximum vertex degree in any biconnected component of the graph. In the *request-delete-model* players are free to delete any bidirectional links and add any directed links, since these additions can be done unilaterally and can be viewed as requests for bidirected links. For this model we give an $O(n^3)$ time algorithm for verifying Nash equilibria in trees. Finally, in the *add–delete-model* we allow a node to make arbitrary deletions and the addition of a single bidirectional link if it would increase the page rank of the other player also. In this model we give a parametric algorithm for verifying Nash equilibria in general graphs and characterize so called α-insensitive Nash Equilibria. We also give a result showing a large class of graphs where there is an edge addition that causes the PageRank of both of its endpoints to increase, suggesting convergence towards complete subgraphs.

1. Introduction

Introduced by Larry Page and Sergey Brin [3], the PageRank of a web page is an important basis of the Google search engine and a very successful application of a mathematical concept in the IT world. PageRank is a value that is assigned to each web page according to the stationary distribution of an α-random walk on the web graph. Here an α-random walk is a random walk modified to make a random jump with probability α at each step and a random jump is a move to a node according to a given distribution vector $\boldsymbol{q}$.

Unlike rankings based on content such as keywords, tags, etc., PageRank focuses solely on the hyperlink structure of the given web graph. Web links themselves possess strategic worth and hence web authors often try to boost the PageRank of their web pages by carefully choosing links to other pages. Since these authors behave strategically in a self-interested way, this is a typical example of a non-cooperative game. In fact, Hopcroft and Sheldon recently introduced the PageRank game as a game theoretic model played over a directed graph [8]. Each player is identified with a node, and a strategy is

* Corresponding author at: School of Computer Science, McGill University, H3A 2A7, Canada. Tel.: +1 5142843195.
E-mail address: avis@cs.mcgill.ca (D. Avis).

0166-218X/$ – see front matter
http://dx.doi.org/10.1016/j.dam.2013.09.022

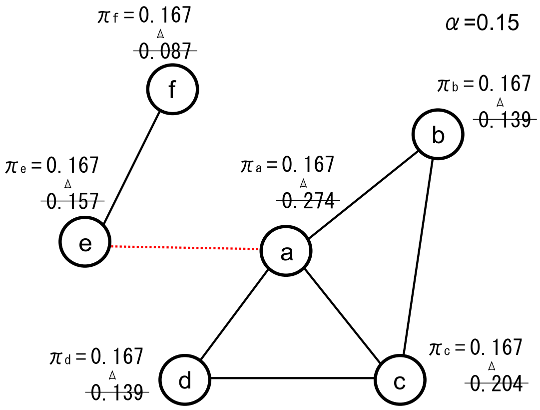

Fig. 1. Edge deletion.

a specification of a set of outlinks to other nodes. The payoff for each player is the PageRank value for their node which is calculated on the resulting directed graph. The obvious goal of each player is to maximize their payoff.

In [8], the authors proved a nice property of this game, namely the best strategy of a player v is to place her outlinks to the nodes u having largest potential value ϕ_{uv}. The potential ϕ_{uv} measures the probability of returning to v before the first jump and does not depend on the outlinks from v if the other nodes do not change their outlinks. Thus, a simple greedy algorithm exists for deciding if a given graph is in Nash equilibrium and a nice characterization of Nash equilibria graphs is possible. Interestingly, it turns out that such graphs representing Nash equilibria have very strong regularity properties (see Section 3 for details). The purpose of this paper is to study similar problems on undirected graphs.

Motivation. Social networks have become one of the defining paradigms of our time, with an enormous influence on how decisions are taken and events unfold. As with web graphs, content by itself will rarely be enough to explain the dynamics of these networks. The underlying graph structure itself surely plays a role in how new relations are formed and old relations broken. In considering two major social networks, Facebook and Twitter, a casual glance shows a radically different graph structure in spite of the fact that they have a comparable number of similar users. Facebook, an undirected graph, has few nodes with degree more than a 1000. Twitter, a directed graph, has nodes (such as Katy Perry) with in-degree 28 million and out degree just 115. A basic difference in the dynamics of the two networks is edge addition, which requires the approval of both nodes in an undirected graph but does not in a directed graph. The ability to add and delete links instantly in a directed network allows for an extremely rapid dynamically changing graph structure. Anecdotal evidence points to a much more stable graph structure in Facebook which apparently consists of large number of relatively small very dense subgraphs.

Another example of an undirected network is the graph of international bilateral agreements between universities. We might consider PageRank as measuring how prestigious a university is, and universities might only accept agreements if it increases their prestige. Finally we might consider the coauthorship graph, possibly one of the oldest social networks, defined so that people could find their “Erdos number”. Here edge deletions are not permitted, but edge additions could conceivably be influenced by the PageRank of the given nodes.

Our motivation is to build models for undirected graphs and to study their dynamics. Our basic tool will be to adopt PageRank as a quantity that users try to optimize. Under this assumption we will study how undirected networks evolve, what networks in equilibrium look like, and contrast this to the case of directed networks. Whether users of these networks actually behave in this way is beyond the scope of this paper.

Outline of the paper. We introduce three different models for PageRank games on undirected graphs. Our study mainly focuses on the *deletion-model* (described in Section 3) where a player cannot create a new link but may unilaterally delete an existing one. In the directed web graph model described above, what v intuitively does is to cut its links to nodes u having smaller ϕ_{uv} values, assuming that u will not delete its edge to v. In the deletion-model, if v cuts its outlink to u we assume that u also cuts its outlink to v either automatically, or as a form of revenge. In Fig. 1 deleting edge *ae* increases *e*’s PageRank but decreases *a*’s. So *e* may unilaterally choose to do this. Note after the edge deletion, if *a* proposed to reinstate the edge *e* would refuse.

Unlike the directed graph model, in the undirected model a node cannot add a new edge by acting unilaterally, but must seek the permission of the other node of the edge. It turns out that the class of equilibria graphs in the deletion-model is larger than in the directed model. Unfortunately, the nice property of the original model that ϕ_{uv} does not depend on the outlinks from v, no longer holds. Hence the greedy algorithm for the Nash decision problem does not work, either, and there seems to be no obvious way of checking the equilibrium condition.

In Section 3.1 we give an $O(n^2)$ time algorithm for the case where the graph is a tree. In Section 3.2 we gave a parametric algorithm for general graphs, where the parameter k is the maximum degree of any vertex in any biconnected component that contains it. Biconnected components roughly correspond to local clusters of web pages, where one could expect the parameter k to be relatively small. Nodes linking biconnected clusters may have arbitrarily large degree without changing the time complexity. We give an $O(2^k n^4)$ time algorithm for general graphs.