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Existence of a pure strategy equilibrium in finite symmetric games where payoff functions are integrally concave

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ABSTRACT

In this paper we show that a finite symmetric game has a pure strategy equilibrium if the payoff functions of players are integrally concave due to Favati and Tardella (1990). Since the payoff functions of any two-strategy game are integrally concave, this generalizes the result of Cheng et al. (2004). A simple algorithm to find a pure strategy equilibrium is also provided.

1. Introduction

The study of symmetric games dates back to Nash's seminal paper (Nash [16]). There the general existence of a mixed strategy equilibrium for any finite game was shown, with an additional information for symmetric games that there exists a *symmetric* mixed strategy equilibrium. Several studies extended this result and examined conditions under which symmetric games have symmetric equilibria, for example, we can refer to the results of Dasgupta and Maskin [5], Reny [17], Becker and Damianov [3], and Amir et al. [1]. These results, however, are concerned with either infinite games or mixed strategy equilibria in finite games and there are few studies about *pure* strategy equilibria in *finite* symmetric games. Cheng et al. [4] showed that every symmetric *two-strategy* game has a (not necessarily symmetric) pure strategy equilibrium, which is also verified by the fact that every symmetric two-strategy game is a potential game (Uno [19]). They also remarked that the generalization to the strategy sets of more than two strategies is generally impossible, quoting the Rock–Paper–Scissors game as a counterexample.

In this paper we show that a symmetric game in which each player's set of strategies is a finite one-dimensional integer interval has a pure strategy equilibrium if the payoff functions of players are *integrally concave* due to Favati and Tardella [6]. The class of integrally concave functions is a class of discrete functions having a feature of continuous concave functions: the local maximum coincides with the global maximum. Since the payoff functions of any two-strategy game are integrally concave, our existence result generalizes the result of Cheng et al. [4].

The results of the present paper are particularly concerned with the following four areas. The first area is the existence of pure strategy equilibria in finite games. A pure strategy equilibrium in a finite game is intuitively appealing in many environments and it is well-known that some classes of finite games, for example, supermodular games (Milgrom and Roberts [9], Milgrom and Shannon [10]) and potential games (Monderer and Shapley [12]) always have a pure strategy

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equilibrium. Confining ourselves to the class of symmetric games, we show, in Section 4, that the class of games with integrally concave payoffs is different from the classes of supermodular games and potential games. Thus the class of finite symmetric games with integrally concave payoffs is a new class of games ensuring the existence of pure strategy equilibria.

The second literature related to the present paper is the symmetry of equilibria in symmetric games. Some results on infinite games (Dasgupta and Maskin [5], Reny [17], Becker and Damianov [3]) showed that symmetric games have symmetric equilibria under some conditions. Asymmetric equilibria and symmetry-breaking also have gathered attentions in recent studies. Amir et al. [1] constructed two general classes of infinite symmetric games which always possess only asymmetric pure strategy equilibria. Fey [7] constructed a two-player symmetric game which has only asymmetric equilibria both in pure and mixed strategies. Focusing on pure strategies in finite games, Amir et al. [2] showed that any pure strategy equilibrium is symmetric in strictly supermodular symmetric games with one-dimensional strategy sets, but asymmetric equilibria are possible in strictly supermodular symmetric games with multi-dimensional strategy sets. In the present paper, we show that a finite symmetric game with integrally concave payoffs possesses either a symmetric equilibrium or an asymmetric equilibrium in pure strategies in which the difference of strategies between any two players is at most one.

Third, our paper is related to discrete convex analysis (Murota [14]) that is recently developed in the field of optimization theory and discrete mathematics. In the theory, several concepts of discrete convexity of functions are proposed in order to extend the usual convex analysis to discrete settings. Integral convexity originated in Favati and Tardella [6] is a weak concept of discrete convexity and covers all the important classes of discrete convexity such as M-convexity (Murota [13]) and $M^{\natural}$-convexity (Murota and Shioura [15]). These latter stronger discrete convexities are required to ensure some properties such as duality and separation which are fundamental in convex optimization (see Murota [14]). Our result shows, however, that integral concavity (the negative of the integral convexity) is sufficient to ensure the existence of pure strategy equilibria in symmetric games. We also show, in Section 4, that the concave extensibility and Miller's discrete concavity, which are weaker than integral concavity, do not imply the existence. The studies of discrete concavity in noncooperative game theory are also found in Ui [18] for potential games, Iimura and Yang [8] for supermodular games (with integrally concave payoffs, similarly to the present paper), and van der Laan et al. [20] for Cournot games of complementary goods.

Finally, an algorithm to find a pure strategy equilibrium provided in the present paper is interesting in view of the complexity of computation. Cheng et al. [4] asserted that the symmetry of a game reduces the burden of the computation to find an equilibrium and proposed some ideas for the computation. Our proof of the existence of a pure strategy equilibrium is constructive and can be applicable to an algorithm for finding a pure strategy equilibrium. Our algorithm finds a pure strategy equilibrium in $O(n+m)$ evaluations of a payoff function, given an n-person symmetric game with m strategies per player.

The paper is organized as follows. Section 2 gives some basic definitions. Section 3 proves our claim, followed by the equilibrium algorithm. In Section 4, we discuss the conditions of our theorem and the relationship of our games to the supermodular games and potential games. Concluding remarks are given in Section 5.

2. Definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{Z}^n$ be the set of its integer points. We denote by e^i the ith unit vector of $\mathbb{R}^n$, $i = 1, \ldots, n$. For any set $X \subseteq \mathbb{R}^n$, $\mathrm{conv}(X)$ denotes the convex hull of X. We say that a set X is an n-dimensional integer interval, if it is written as $X = \{z \in \mathbb{Z}^n \mid a_i \le z_i \le b_i \ \forall i = 1, \ldots, n\}$ with some $a_i, b_i \in \mathbb{Z}$ such that $a_i < b_i$ for $i = 1, \ldots, n$. Let $\mathrm{N}(y) := \{z \in \mathbb{Z}^n \mid |z_i - y_i| < 1 \ \forall i = 1, \ldots, n\}$.

Definition 2.1. Let X be an n-dimensional integer interval and $f : X \to \mathbb{R}$ a discrete function. The *local concave extension* of f is a piecewise-linear function $\tilde{f} : \mathrm{conv}(X) \to \mathbb{R}$ defined for each $y \in \mathrm{conv}(X)$ (using the points $z \in \mathrm{N}(y)$) by

$$\tilde{f}(y) := \max\left\{ \sum_{z\in \mathrm{N}(y)} \alpha_z f(z) \;\middle|\; \sum_{z\in \mathrm{N}(y)} \alpha_z z = y,\ \sum_{z\in \mathrm{N}(y)} \alpha_z = 1,\ \alpha_z \ge 0 \text{ for all } z \in \mathrm{N}(y) \right\}.$$

Note that the restriction of the local concave extension $\tilde{f}$ to any unit cube in $\mathrm{conv}(X)$ is concave, but $\tilde{f}$ is not necessarily concave on the entire domain.

Definition 2.2 (*Favati and Tardella [6]*)**.** Let X be an n-dimensional integer interval and $f : X \to \mathbb{R}$ a discrete function. f is *integrally concave* if its local concave extension is concave on $\mathrm{conv}(X)$.

A game is a three-tuple $(N, (S_i)_{i\in N}, (P_i)_{i\in N})$, where $N := \{1, \ldots, n\}$ $(n \ge 2)$ is the set of players, S_i is the set of strategies of $i \in N$, and P_i is the payoff function of $i \in N$ defined on the set of strategy profiles $S := S_1 \times \cdots \times S_n$. A game is finite if all the strategy sets are finite sets. We assume throughout the present paper that the set of strategies of every player is an identical one-dimensional integer interval, i.e., we assume that

$$S_1 = \cdots = S_n = \{z \in \mathbb{Z} \mid a \le z \le b\}$$

for some $a, b \in \mathbb{Z}$ such that $a < b$. Note that the set of strategy profiles is an n-dimensional integer interval. According to Nash [16], we define the symmetry of a game as follows. Let Π be the set of all the bijections $\pi : N \to N$. We call any $\pi \in \Pi$