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Some advances on the set covering polyhedron of circulant matrices

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ABSTRACT

Studying the set covering polyhedron of consecutive ones circulant matrices, Argiroffo and Bianchi found a class of facet defining inequalities, induced by a particular family of circulant minors. In this work we extend these results to inequalities associated with every circulant minor. We also obtain polynomial separation algorithms for particular classes of such inequalities.

1. Introduction

The well-known concept of domination in graphs was introduced by Berge [6] in 1962, modeling many utility location problems in operations research.

Given a graph $G = (V, E)$ a *dominating set* is a subset $D \subset V$ such that every node outside D is adjacent to at least one node in D. Given a cost vector $w \in \mathbb{R}^{|V|}$, the *Minimum Weight Dominating Set Problem* (MWDSP for short), consists of finding a dominating set D such that $\sum_{v \in D} w_v$ is minimum. MWDSP arises in many applications, involving the strategic placement of men or pieces on the nodes of a network. As an example, consider a computer network in which one wishes to choose a smallest set of computers that are able to transmit messages to all the remaining computers [18]. Many other interesting examples include sets of representatives, school bus routing, (r, d)-configurations, radio stations, social network theory, kernels of games, etc. [15].

The MWDSP is NP-hard for general graphs and has been extensively investigated from an algorithmic point of view ([7,11,12,14] among others). The cardinality version (that is when the weights are 0 and 1) has been shown to be polynomially solvable in several classes of graphs such as cactus graphs [16] and the class of series–parallel graphs [17].

However, a few results on the MWDSP derived from the polyhedral point of view are known. An interesting result in this context can be found in [10], working on the problem when the underlying graph is a cycle.

Actually, the MWDSP corresponds to particular instances of the Minimum Weighted Set Covering Problem (MWSCP).

Indeed, given an $m \times n$ 0, 1 matrix A, a *cover of* A is a vector $x \in \{0, 1\}^n$ such that $Ax \geq \mathbf{1}$, where $\mathbf{1}$ is the vector with all components at value one. Given a cost function $w \in \mathbb{R}^n$, the MWSCP consists of solving the integer program

$$\min\{wx : Ax \geq \mathbf{1},\ x \in \{0, 1\}^n\}.$$

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This is equivalent to solving the problem

$$\min\{wx : x \in Q^*(A)\}$$

where $Q^*(A)$ is the convex hull of points in $\{x \in \{0, 1\}^n : Ax \geq \mathbf{1}\}$. The set $Q^*(A)$ is usually called the *set covering polyhedron* associated with A.

In particular, given a graph $G = (V, E)$, if A is a matrix such that each row corresponds to the characteristic vector of the closed neighborhood of a node $v \in V$ (i.e., A is the *closed neighborhood matrix of* G) then every cover of A is the characteristic vector of a dominating set of G and conversely. Therefore, solving the MWSCP on A is equivalent to solve the MWDSP on G.

It is easy to see that the closed neighborhood matrix of a cycle is a *circulant* matrix. Hence, the findings in [10] correspond to obtaining the complete description of the set covering polyhedron for the 0, 1 $n \times n$ matrices having three consecutive ones per row, known as the family of circulant matrices C_n^3.

In general, the closed neighborhood of a *web* graph is a circulant matrix. Web graphs have been thoroughly studied in the literature (see [21,23,24]).

The main goal of this work is the study of the MWSCP on circulant matrices and its direct consequences on the MWDSP when the underlying graph is a web graph.

Previous results on the set covering polyhedron of circulant matrices can be found in [2,3,13,19,20]. In [13] it was observed that if A is a circulant matrix then every set $\{x \in [0, 1]^n : Ax \geq \mathbf{1},\ x_i = 1\}$ for $i = 1, \ldots, n$ is an integer polyhedron. Then it holds that solving the MWSCP on a circulant matrix can be thought as solving at most n linear programs. Hence, the MWSCP on circulant matrices results in a polynomial problem.

In Section 2 of this work, we present basic definitions and preliminaries needed for the remaining sections. In Section 3 we introduce a family of valid inequalities for the set covering polyhedron of circulant matrices. We obtain sufficient conditions for a valid inequality to define a facet of the polyhedron. We also conjecture that this condition is also necessary. In Section 4 we prove that a subfamily of the inequalities presented in Section 3 can be separated in polynomial time.

A preliminary version of this work appeared without proofs in [8].

2. Definitions, notations and preliminary results

In what follows, every time we state $S \subset \mathbb{Z}_n$ for some $n \in \mathbb{N}$, we consider $S \subset \{0, \ldots, n-1\}$ and the addition between the elements of S is taken modulo n.

Given a set F of vectors in $\{0, 1\}^n$, we say $y \in F$ is a dominating vector (of F) if there exist $x \in F$ such that $x \leq y$. It can be also said that x is dominated by y.

From now on, every matrix has 0,1 entries, no zero columns and no dominating rows. If A is such an $m \times n$ matrix, its rows and columns are indexed by $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. Two matrices A and A' are *isomorphic* and we write $A \approx A'$, if A' can be obtained from A by permutation of rows and columns.

If $S \subset \mathbb{Z}_m$ and $T \subset \mathbb{Z}_n$, let $A_{S,T}$ be the submatrix of A with entries a_{ij} where $i \in S$ and $j \in T$.

Given $N \subset \mathbb{Z}_n$, let us denote by $R(N) = \{j \in \mathbb{Z}_m : j \text{ is a dominating row of } A_{\mathbb{Z}_m, \mathbb{Z}_n - N}\}$. A *minor of* A *obtained by contraction of* N, denoted by A/N, is the matrix $A_{\mathbb{Z}_m - R(N), \mathbb{Z}_n - N}$. In this work, when we refer to *a minor* of A we are always considering a minor obtained by contraction.

Observe that there exists a one-to-one correspondence between a vector $x \in \{0, 1\}^n$ and the subset $S_x \subset \mathbb{Z}_n$ whose characteristic vector is x itself. Hence, we agree to abuse of notation by writing x instead of S_x. In this way, if $x \in \{0, 1\}^n$, we write $i \in x$ meaning that $x_i = 1$. Also, if x is dominated by $y \in \{0, 1\}^n$ then we write $x \subset y$.

Remind that a *cover* of a matrix A is a vector $x \in \{0, 1\}^n$ such that $Ax \geq \mathbf{1}$. In addition, the *cardinality* of a cover x is denoted by $|x|$ and equals $\mathbf{1}x$. A cover x is *minimum* if it has the minimum cardinality and in this case $|x|$ is called the *covering number* of the matrix A, denoted by $\tau(A)$. Observe that every cover of a minor of A is a cover of A and then, for all $N \subset \mathbb{Z}_n$, it holds that $\tau(A/N) \geq \tau(A)$.

Recall that the set covering polyhedron of A, denoted by $Q^*(A)$, is defined as the convex hull of its covers. The polytope $Q(A) = \{x \in [0, 1]^n : Ax \geq \mathbf{1}\}$ is known as the *linear relaxation of* $Q^*(A)$. When $Q^*(A) = Q(A)$ the matrix A is ideal and the MWSCP can be solved in polynomial time (in the size of A).

Given n and k with $2 \leq k \leq n-2$, for every $i \in \mathbb{Z}_n$ let $C^i = \{i, i+1, \ldots, i+(k-1)\} \subset \mathbb{Z}_n$. The *circulant* matrix C_n^k is the square matrix whose i-th row is the incidence vector of C^i. Observe that, for $j \in \mathbb{Z}_n$, the j-th column of C_n^k is the incidence vector of C^{j-k+1}.

We say that a minor of C_n^k is a *circulant minor* if it is isomorphic to a circulant matrix.

Remark 1. Let C_n^k be a circulant matrix and let $x = \{i_j : j \in \mathbb{Z}_r\} \subset \mathbb{Z}_n$ with $0 \leq i_0 < i_1 < \cdots < i_{r-1} \leq n-1$. The following propositions are equivalent:

(i) x is a cover of C_n^k,
(ii) $i_{j+1} - 1 \in C^{i_j}$ for all $j \in \mathbb{Z}_r$,
(iii) $i_{j-1} \in C^{i_j - k}$ for all $j \in \mathbb{Z}_r$.