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Neighbor sum distinguishing edge colorings of graphs with bounded maximum average degree

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ABSTRACT

A proper $[k]$-edge coloring of a graph G is a proper edge coloring of G using colors of the set $[k]$, where $[k] = \{1, 2, \ldots, k\}$. A neighbor sum distinguishing $[k]$-edge coloring of G is a proper $[k]$-edge coloring of G such that, for each edge $uv \in E(G)$, the sum of colors taken on the edges incident with u is different from the sum of colors taken on the edges incident with v. By $ndi_{\sum}(G)$, we denote the smallest value k in such a coloring of G. The average degree of a graph G is $\frac{\sum_{v\in V(G)} d(v)}{|V(G)|}$; we denote it by $ad(G)$. The maximum average degree $mad(G)$ of G is the maximum of average degrees of its subgraphs. In this paper, we show that, if G is a graph without isolated edges and $mad(G) \le \frac{5}{2}$, then $ndi_{\sum}(G) \le k$, where $k = \max\{\Delta(G) + 1, 6\}$. This partially confirms the conjecture proposed by Flandrin et al.

1. Introduction

The terminology and notation used but undefined in this paper can be found in [5]. Let $G = (V, E)$ be a graph. We use the symbols $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$ to denote the vertex set, edge set, maximum degree, and minimum degree of G, respectively. Let $d_G(v)$, or simply $d(v)$, denote the degree of a vertex v in G. A vertex v is called a *k-vertex* (respectively, *k^--vertex*), if $d(v) = k$ (respectively, $d(v) \le k$). A vertex v is called a *leaf* of G if $d(v) = 1$. If $d(v) = 2$ and the two neighbors of v are a d_1-vertex and a d_2-vertex, respectively, vertex v is called a *(d_1, d_2)-vertex*. Similarly, we can define a (d_1, d_2^-)-vertex and a (d_1, d_2^+)-vertex. The *girth* of a graph G is the length of a shortest cycle in G, and we denote it by $g(G)$. The *average degree $ad(G)$* of a graph G is $\frac{\sum_{v\in V(G)} d(v)}{|V(G)|}$. The *maximum average degree $mad(G)$* of G is the maximum of the average degrees of its subgraphs.

Let $[k]$ be a set of colors, where $[k] = \{1, 2, \ldots, k\}$, and let c be an edge coloring of G for which $c : E(G) \to [k]$. By $w(v)$ (respectively, $S(v)$), we denote the sum (respectively, set) of colors taken on the edges incident with v; i.e., $w(v) = \sum_{uv\in E(G)} c(uv)$ (respectively, $S(v) = \{c(uv)|uv \in E(G)\}$). If the coloring c is proper, then we call the coloring c such that $w(v) \ne w(u)$ (respectively, $S(u) \ne S(v)$) for each edge $uv \in E(G)$ a *neighbor sum distinguishing* (respectively, *neighbor distinguishing*) *$[k]$-edge coloring* of G. If c is not assumed to be proper, then the coloring c such that $w(v) \ne w(u)$ is called *vertex-coloring $[k]$-edge-weighting*. By $ndi_{\sum}(G)$ (respectively, $ndi(G)$), we denote the smallest value k such that G has a neighbor sum distinguishing (respectively, *neighbor distinguishing*) $[k]$-edge coloring of G.

In 2004, Karoński et al. [12] introduced the notion of vertex-coloring $[k]$-edge-weighting and brought forward the following conjecture.

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Conjecture 1 (*1–2–3-Conjecture [12]*). *Every graph without isolated edges admits a vertex-coloring 3-edge-weighting.*

Addario-Berry et al. [1] showed that every graph without isolated edges admits a vertex-coloring 30-edge-weighting. This bound was improved to 16 by Addario-Berry et al. [2], and later improved to 13 by [21]. Recently, Kalkowski et al. [11] showed that every graph without isolated edges admits a vertex-coloring 5-edge-weighting.

We know that, to have a neighbor sum distinguishing (neighbor distinguishing) coloring, G cannot have an isolated edge (we call it normal). Apparently, for any normal graph G, $ndi(G) \leq ndi_{\Sigma}(G)$. In 2002, Zhang et al. [22] proposed the following conjecture.

Conjecture 2 (*[11]*). *If G is a normal graph with at least six vertices, then $ndi(G) \leq \Delta(G) + 2$.*

Balister et al. [4] proved Conjecture 2 for bipartite graphs and for graphs G with $\Delta(G) = 3$. If G is bipartite planar with maximum degree $\Delta(G) \geq 12$, Conjecture 2 was confirmed by Edwards et al. [8]. Hatami [10] showed that, if G is a normal graph and $\Delta(G) > 10^{20}$, then $ndi(G) \leq \Delta(G) + 300$. Akbari et al. [3] proved that $ndi(G) \leq 3\Delta(G)$ for any normal graph. Bu et al. [6] proved Conjecture 2 for planar graphs of girth at least 6. Wang et al. [18,19] confirmed Conjecture 2 for sparse graphs and K_4-minor free graphs. More precisely, in [18], the authors showed that, if G is a normal graph, $mad(G) < \frac{5}{2}$, then $ndi(G) \leq \Delta(G) + 1$ and $ndi(G) = \Delta(G) + 1$ if and only if G has two adjacent $\Delta(G)$-vertices.

Recently, Flandrin et al. [9] studied the neighbor sum distinguishing colorings for cycles, trees, complete graphs, and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 3 (*[9]*). *If G is a connected graph on at least three vertices and $G \neq C_5$, then $\Delta(G) \leq ndi_{\sum}(G) \leq \Delta(G) + 2$.*

Flandrin et al. [9] gave an upper bound for each connected graph G with maximum degree $\Delta \geq 2$.

Theorem 1.1 (*[9]*). *For each connected graph with maximum degree $\Delta \geq 2$, we have $ndi_{\sum}(G) \leq \lceil \frac{7\Delta-4}{2} \rceil$.*

Wang and Yan [20] improved this bound to $\lceil \frac{10\Delta(G)+2}{3} \rceil$. Recently, Przybyło [16] proved that $nsdi(G) \leq 2\Delta(G) + col(G) - 1$, where $col(G)$ is the *coloring number* of G, which is defined as the least integer k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbors; hence $col(G) - 1 \leq \Delta(G)$, and thus $2\Delta(G) + col(G) - 1 \leq 3\Delta(G)$. Dong et al. [7] considered the neighbor sum distinguishing colorings of planar graphs and showed the following result.

Theorem 1.2 (*[7]*). *If G is a normal planar graph, then $ndi_{\sum}(G) \leq \max\{2\Delta(G) + 1, 25\}$.*

Later, Wang et al. [17] improved this bound to $\max\{\Delta(G) + 10, 25\}$. In this paper, we will prove the following results.

Theorem 1.3. *Let G be a normal graph. If $mad(G) \leq \frac{5}{2}$, then $ndi_{\sum}(G) \leq k$, where $k = \max\{\Delta(G) + 1, 6\}$.*

Corollary 1.1. *Let G be a normal graph. If $mad(G) \leq \frac{5}{2}$, $\Delta(G) \geq 5$, then $ndi_{\sum}(G) \leq \Delta(G) + 1$.*

It is well known that, if G is a planar graph with girth g, then $mad(G) < \frac{2g}{g-2}$, so the following corollary is obvious.

Corollary 1.2. *Let G be a normal planar graph. If $g(G) \geq 10$ and $\Delta(G) \geq 5$, then $ndi_{\sum}(G) \leq \Delta(G) + 1$.*

Note that, if G contains two adjacent vertices of maximum degree, then $ndi_{\sum}(G) \geq \Delta(G) + 1$. So the bound $\Delta(G) + 1$ in Corollary 1.1 is sharp. Furthermore, Corollary 1.1 implies a result of Wang et al. [18] on the neighbor distinguishing coloring of sparse graphs. For neighbor sum distinguishing total colorings, see [13–15].

2. Preliminaries

First, we give some lemmas.

Lemma 2.1 (*[9]*). *If $m = 0 \pmod 3$, then $ndi_{\sum}(C_m) = 3$; otherwise, $ndi_{\sum}(C_m) = 4$.*

In the following lemmas, all the elements in each set are positive integers.

Lemma 2.2. *Let S_1, S_2 be two sets, and let $S_3 = \{a + b \mid a \in S_1,\ b \in S_2,\ a \neq b\}$. If $|S_1| = k,\ k \geq 3$, $|S_2| = 2$, then $|S_3| \geq k$.*

Proof. Let $S_1 = \{x_1, \ldots, x_k\}, x_1 < \cdots < x_k$, $S_2 = \{m, M\},\ m < M$. Clearly, the set $(\{x_1 + m, \ldots, x_k + m\} \setminus \{2m\}) \cup \{z\}$, where $z = x_k + M$ if $x_k \neq M$, $z = x_{k-1} + M$ if $x_{k-1} \neq m$ and $x_k = M$, and $z = x_{k-2} + M$ if $x_{k-1} = m$ and $x_k = M$, has at least k elements. □

Lemma 2.3. *Let S_1, S_2 be two sets, and let $S_3 = \{\alpha + \beta \mid \alpha \in S_1, \beta \in S_2, \alpha \neq \beta\}$. If $|S_1| = |S_2| = 2$ and $S_1 \neq S_2$, then $|S_3| \geq 3$.*

Proof. Since $S_1 \neq S_2$, we have $|S_1 \cap S_2| \leq 1$. If $|S_1 \cap S_2| = 1$, then we assume that $S_1 = \{x_1, x_2\}$ and $S_2 = \{x_1, y\}$. Clearly, $\{x_1 + y, x_1 + x_2, x_2 + y\} \subseteq S_3$. Hence $|S_3| \geq 3$. Otherwise, $|S_1 \cap S_2| = 0$. Without loss of generality, let $S_1 = \{x_1, x_2\}$, $S_2 = \{y_1, y_2\}$ such that $x_1 < x_2, y_1 < y_2$. Then $\{x_1 + y_1, x_1 + y_2, x_2 + y_2\} \subseteq S_3$. So we have $|S_3| \geq 3$. □

The following lemma is obvious, so we omit the proof.

Lemma 2.4. *Let S be a set of size $k + 1$. Let $S_1 = \{\sum_{i=1}^{k} x_i \mid x_i \in S, x_i \neq x_j,\ \text{if}\ i \neq j, 1 \leq i, j \leq k\}$. Then $|S_1| \geq k + 1$.*