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Dense bipartite circulants and their routing via rectangular twisted torus

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ABSTRACT

Whereas the maximum number of vertices in a four-regular circulant of diameter a is $2a^2 + 2a + 1$, the bound turns out to be $2a^2$ if the condition of bipartiteness is imposed. Tzvieli presented a family of $\psi(a)$ such dense bipartite circulants on $2a^2$ vertices for each $a \geq 3$, where $\psi(a)$ denotes the number of positive integers less than or equal to $\lfloor \frac{1}{2}(a-1) \rfloor$ that are coprime with a. The present paper shows that each of those graphs is obtainable from the $2a \times a$ rectangular twisted torus by appropriately trading a maximum of $2a$ edges for as many new edges. The underlying structural similarity between the two graphs leads to a simple intuitive routing algorithm for the circulants. The result closely parallels the routing in dense nonbipartite circulants on $2a^2 + 2a + 1$ vertices. Additional results include a set of vertex-disjoint paths between every pair of distinct vertices in the circulants, and a proof that the $2a \times a$ rectangular twisted torus itself is probably not a circulant.

1. Introduction and preliminaries

The *circulant graphs*, which we formally define below, constitute a subfamily of Cayley graphs [16]. They possess a number of attractive properties that render them fit for applications in areas such as distributed systems, VLSI, parallel machines and perfect codes [4,6,7,22,23]. Some of their welcome features include high connectivity, high symmetry and a rich cycle structure, which together lead to a very high degree of fault tolerance. Accordingly, they have been a topic of extensive research in engineering, computer science and mathematics for long [5,16,29].

Routing is the process of selecting path(s) in a graph/network along which to send the traffic [26]. It is a vital component of all networks. Unfortunately, the associated problem of finding a shortest path between two vertices in a circulant is NP-hard [9]. This is also because the graph admits a very compact representation leading to a small input size. A related aberration is that $dia\big(\mathcal{C}_n(1,\ s)\big)$ is not monotone in n for a fixed s [12]. This is to be contrasted with the hypercube and the torus in which a shortest path is traceable fairly quickly and intuitively. Not surprisingly, the problem of routing is crucial in the study of circulants [11,32].

For a given positive integer $a \geq 3$, a four-regular circulant of diameter a may have a maximum of $2a^2+2a+1$ vertices [3]. Indeed, such a dense circulant (which is necessarily nonbipartite) is known to exist for a long time, and it has been a topic of deep research [3–5,16,19]. See Martinez et al. [25] for recent results on its routing algorithms and implementation issues.

If the condition of bipartiteness is imposed, then the maximum number of vertices in a four-regular circulant of diameter a turns out to be $2a^2$ [14]. Again, such dense bipartite circulants are known to exist. In particular, Tzvieli [30] presented a family of $\psi(a)$ bipartite circulants on $2a^2$ vertices for each $a \geq 3$, where $\psi(a)$ denotes the number of positive integers less than or equal to $\lfloor \frac{1}{2}(a-1) \rfloor$ that are coprime with a. A major finding of this paper is that each such graph is obtainable from the $2a \times a$ *rectangular twisted torus* (RTT), which we formally define below, by appropriately trading a maximum of $2a$ edges for as many new edges. That, in turn, leads to a simple intuitive *routing algorithm* in respect of those graphs.

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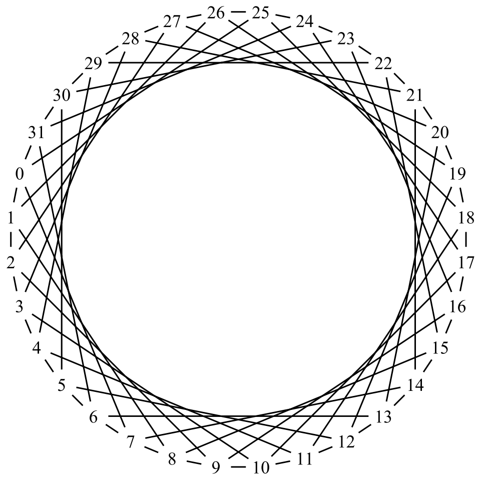

Fig. 1. The circulant $\mathcal{C}_{32}(1,\ 7)$.

Additional results include a set of vertex-disjoint paths between every pair of distinct vertices in the circulants, and a proof that the $2a \times a$ rectangular twisted torus itself is probably not a circulant.

In a closely related study, Beivide et al. [3] earlier presented a four-regular circulant $\mathcal{C}_n(b-1, b)$ on n vertices for each $n \geq 6$, where $b = \lceil\sqrt{n/2}\rceil$. The graph, known also as a *midimew network*, is such that its diameter and the average (shortest) distance are the least among all four-regular graphs on as many vertices. It turns out that the circulants in the present paper share those characteristics, hence they may be viewed as their bipartite counterparts.

1.1. Basic definitions

When we speak of a graph G, we mean a finite, simple, undirected and connected graph. Let $dist(u, v)$ denote the (shortest) distance between u and v, where the underlying graph will be clear from the context. Next, let $dia(G)$ represent the *diameter* of G, i.e., the maximum of the distances between any two vertices in G. Vertices that are at a distance of $dia(G)$ from u are called *diametrical* relative to u. We employ the terms *vertex* and *node* as synonyms. For missing details, see Hammack et al. [13].

Let P_m denote a *path* on m vertices, and C_n a *cycle* on n vertices, where $m \geq 1$ and $n \geq 3$. The *Cartesian product* $G\Box H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows: $V(G\Box H) = V \times W$ and $E(G\Box H) = \{\{(a, x), (b, y)\}: \{a, b\} \in E$ and $x = y$, or $\{x, y\} \in F$ and $a = b\}$. See Fig. 2(i) for $P_8\Box P_4$.

Let n, r and s be positive integers, where $n \geq 6$, and $1 \leq r < s < \lfloor n/2 \rfloor$. A *four-regular circulant graph* $\mathcal{C}_n(r, s)$ consists of the vertex set $\{0, \ldots, n-1\}$ and the edge set $\{\{i,\ i \pm r\}, \{i,\ i \pm s\} : 0 \leq i \leq n-1\}$, where $i \pm r$ and $i \pm s$ are each modulo n. A circulant $\mathcal{C}_n(r, s)$ with $r = 1$ is also known as a chordal ring or a double-loop graph in the literature [14,28]. Meanwhile $\mathcal{C}_{32}(1,\ 7)$ appears in Fig. 1, and Proposition 1.1 states some well-known characteristics of a four-regular circulant.

Proposition 1.1 (*[27,15]*).

1. $\mathcal{C}_n(r,\ s)$ *is connected if and only if* $\gcd(n, r, s) = 1$.
2. $\mathcal{C}_n(r,\ s)$ *is bipartite if and only if* n *is even, and* r *and* s *are both odd.*
3. $\mathcal{C}_n(r,\ s)$ *is isomorphic to* $\mathcal{C}_n(r,\ n-s)$.
4. *If* $\gcd(r, n) = 1$*, then* $\mathcal{C}_n(r, s)$ *is isomorphic to* $\mathcal{C}_n(1, r^{-1}s \mod n)$*, where* r^{-1} *is the multiplicative inverse of* r *relative to* n.
5. *If* $\gcd(t, n) = 1$*, then* $\mathcal{C}_n(r,\ s)$ *is isomorphic to* $\mathcal{C}_n(rt \mod n, st \mod n)$. ■

1.2. Rectangular twisted torus

The $2a \times a$ RTT is an alternative to the $2a \times a$ *torus*, the latter being representable as $C_{2a}\Box C_a$. The vertex set of the RTT is given by $\{(i,\ j) :\ 0 \leq i \leq 2a-1$ and $0 \leq j \leq a-1\}$, while the edge set consists of the following:

- $\{(i, j), (i+1, j)\}$: $0 \leq i \leq 2a-2$ and $0 \leq j \leq a-1$, called the "horizontal" edges
- $\{(i, j), (i, j+1)\}$: $0 \leq i \leq 2a-1$ and $0 \leq j \leq a-2$, called the "vertical" edges
- $\{(0, j), (2a-1, j)\}$: $0 \leq j \leq a-1$, called the "wrap-around" edges, and
- $\{(i, 0), (i+a, a-1)\}$: $0 \leq i \leq 2a-1$, called the "twisted" edges

where the arithmetic is modulo $2a$ in the first coordinate and modulo a in the second. This graph is bipartite, four-regular and nonplanar. Further, its diameter is equal to a [10], and it is obtainable from $P_{2a}\Box P_a$ by introducing the "wrap-around" edges and the "twisted" edges. Graphs $P_8\Box P_4$ and 8×4 RTT appear in Fig. 2.