Discrete Applied Mathematics 166 (2014) 170–177

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Precoloring extension involving pairs of vertices of small distance

Chihoko Ojima, Akira Saito*, Kazuki Sano

Department of Information Science, Nihon University, Sakurajosui 3–25–40, Setagaya-Ku, Tokyo 156–8550, Japan

ARTICLE INFO

Article history:
Received 9 February 2013
Received in revised form 1 September 2013
Accepted 6 October 2013
Available online 1 November 2013

Keywords:
Coloring
Precoloring
Distance

ABSTRACT

In this paper, we consider coloring of graphs under the assumption that some vertices are already colored. Let G be an r-colorable graph and let $P \subset V(G)$. Albertson (1998) has proved that if every pair of vertices in P has distance at least four, then every $(r+1)$-coloring of $G[P]$ can be extended to an $(r+1)$-coloring of G, where $G[P]$ is the subgraph of G induced by P. In this paper, we allow P to have pairs of vertices of distance at most three, and investigate how the number of such pairs affects the number of colors we need to extend the coloring of $G[P]$. We also study the effect of pairs of vertices of distance at most two, and extend the result by Albertson and Moore (1999).

1. Introduction

Graph coloring has a number of applications. One example is a job scheduling problem. In this problem, each job is represented by a vertex, and a pair of vertices is joined by an edge if the corresponding jobs cannot be processed concurrently. In this model, an independent set represents a set of jobs which can be preformed at the same time, and if we assume that each job is processed in a unit time, the chromatic number gives the minimum amount of time in which we can finish all the jobs in the concurrent environment.

In the real world, however, the job scheduling may not be tackled from scratch. In many cases, the schedule of some jobs are already fixed and cannot be changed. In graph coloring, it corresponds to a situation in which some vertices are already colored. A precoloring extension is a problem to handle this situation. In this problem, a graph G, a set of vertices $P \subset V(G)$ and a coloring $d: P \to \boldsymbol{Z}$ of $G[P]$ are given, where $G[P]$ is the subgraph of G induced by P. We call d a *precoloring*. Our task is to find a coloring $V(G) \to \boldsymbol{Z}$ of G whose restriction into P coincides with d. If P is sufficiently sparse, we may expect to extend d to a coloring of G with a few extra colors. For the measure of sparseness, Albertson [1] and Albertson and Moore [5] have considered the minimum distance between the vertices in P.

Theorem A ([1]). *Let G be a graph with chromatic number at most r, and let $P \subset V(G)$. Suppose every pair of distinct vertices in P has distance at least four. Then every $(r+1)$-coloring of P can be extended to an $(r+1)$-coloring of G.*

Theorem B ([5]). *Let G be a graph with chromatic number at most r and let $P \subset V(G)$. Suppose every pair of distinct vertices in P has distance at least three. Then every $(r+1)$-coloring of P can be extended to a $\lceil \frac{3r+1}{2} \rceil$-coloring of G.*

These theorems give insight into the relationship between the distance of precolored vertices and the number of colors necessary to extend the precoloring to a coloring of the whole graph.

On the other hand, again in the real world, the assumptions of Theorems A and B may be idealistic. For example, while we have Theorem A, we may have to deal with a set P of precolored vertices which contains pairs of vertices of distance

* Corresponding author. Tel.: +81 353179342; fax: +81 353179770.
E-mail address: asaito@chs.nihon-u.ac.jp (A. Saito).

0166-218X/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.dam.2013.10.012

three. In this case, we might be forced to use more than $r+1$ colors to extend the given precoloring. But if the number of the pairs of distance three is sufficiently small, we expect that the number of additional colors is also small. Theorem A does not answer this question.

Motivated by this observation, we investigate the situation in which the set of precolored vertices contains pairs of distance at most three and two, and investigate how these pairs affect the conclusion of Theorems A and B, respectively.

For a graph G, $P \subset V(G)$ and a positive integer k, we define $\mathcal{D}_G(P,k)$ by

$$\mathcal{D}_G(P,k) = \big\{\{x,y\} \subset P : x \neq y \text{ and } d_G(x,y) \leq k\big\},$$

where $d_G(x,y)$ is the distance between x and y in G.

In the next section, we give an upper bound to the number of additional colors to extend a given precoloring of P to a coloring of G, which is described in terms of $|\mathcal{D}_G(P,3)|$. In Section 3, we give another bound, which is described in terms of $|\mathcal{D}_G(P,2)|$. In Section 4, we give some concluding remarks.

We remark that this paper is neither the only nor the first one to extend Theorems A and B. The problem of extending a precoloring to the entire graph has been studied in many papers. We refer the readers who are interested in this problem to [2–4,6,7,10–12,14–18]. In particular, Albertson and Hutchinson [3] and Hutchinson and Moore [12] have considered the situation in which the set of precolored vertices induces a graph with several components, and studied distance conditions among these components that guarantee the extension without using an additional color. They have given best-possible results in many cases.

For graph-theoretic notation and definitions not explained in this paper, we refer the reader to [9]. Let G be a graph. Then we denote by $\Delta(G)$ and $\chi(G)$ the maximum degree and the chromatic number of G, respectively. For $x \in V(G)$, we denote the neighborhood of x in G by $N_G(x)$. In this paper, we often deal with the closed neighborhood of G, which is denoted and defined by $N_G[x] = N_G(x) \cup \{x\}$. If $A, B \subset V(G)$ and $A \cap B = \emptyset$, we define $E_G(A,B)$ by $E_G(A,B) = \{ab \in E(G) : a \in A \text{ and } b \in B\}$.

Let $P \subset V(G)$. As we have already seen, a coloring of $G[P]$ is called a precoloring of P in G. In this paper, we always perceive a coloring of G as a mapping $f : V(G) \to \boldsymbol{Z}$. If $d : P \to \boldsymbol{Z}$ is a precoloring of P in G and $f : V(G) \to \boldsymbol{Z}$ is a coloring of G with $f(v) = d(v)$ for every $v \in P$, we say that f *extends* d. For a positive integer r, we denote the set $\{1, 2, \ldots, r\}$ by $[r]$. An r-coloring of G is a coloring of G which uses at most r colors. In this paper, an r-coloring is often perceived as a function from $V(G)$ to $[r]$. For $t \in [r]$, $f^{-1}(t)$ is the set of vertices that receive the color t. We call it the *color class* of $V(G)$ with respect to the color t.

If $e = uv$ is an edge of a graph G, we denote $\{u,v\}$ by $V(e)$. Moreover, for $F \subset E(G)$, we write $V(F)$ for $\bigcup_{e \in F} V(e)$. A matching of G is a set of independent edges in G. Hence if M is a matching, then the size of M, denoted by $|M|$, is the number of edges in M, and $|V(M)| = 2|M|$. If M is a maximum matching of G, $|V(G)| - |V(M)|$ is called the deficiency of G. Concerning the deficiency of a graph, Berge's Formula is well-known. We denote by $o(G)$ the number of components of odd order in G.

Theorem C (*Berge's Formula [8]*)**.** *For a graph G, the deficiency of G is given by* $\max\{o(G-S) - |S| : S \subset V(G)\}$.

A matching M in G is called a perfect matching if $V(M) = V(G)$, and M is called an almost perfect matching if $|V(M)| = |V(G)| - 1$.

2. Pairs of vertices of distance three

In this section, we investigate the effect of the number of vertices which are of distance at most three. Theorem A states that for a graph G with $\chi(G) \leq r$ and $P \subset V(G)$ with $\mathcal{D}_G(P,3) = \emptyset$, every $(r+1)$-coloring of P extends to an $(r+1)$-coloring of G. If $\mathcal{D}_G(P,3) \neq \emptyset$, we may need more than $r+1$ colors. The purpose of this section is to prove that for $t = |\mathcal{D}_G(P,3)|$, $r + O(\sqrt{t})$ colors suffice.

Theorem 1. *Let k be a positive integer. Let G be a graph with $\chi(G) \leq r$ and let $P \subset V(G)$. Suppose $|\mathcal{D}_G(P,3)| \leq \frac{1}{2}k(k+1)$. Then for each precoloring $d : P \to [r+1]$ in G, there exists a coloring $f : V(G) \to [r+k]$ with $f(u) = d(u)$ for each $u \in P$.*

We prove several lemmas to give a proof to Theorem 1. The following lemma has already been proved in [1]. But we tailor its statement so that it fits the subsequent arguments. For the completeness of the paper, we give a proof to it. For two colorings f, g of a graph G, we define $X(f,g)$ by $X(f,g) = \{v \in V(G) : f(v) \neq g(v)\}$.

Lemma 2 (*[1]*)**.** *Let G be a graph with an r-coloring $c : V(G) \to [r]$. Let P be a set of vertices of G with $\mathcal{D}_G(P,3) = \emptyset$. Then for every precoloring $d : P \to [r+1]$, there exists an $(r+1)$-coloring $f : V(G) \to [r+1]$ of G such that*

(1) *$f(x) = d(x)$ for every $x \in P$, and*
(2) *for each $v \in X(c,f)$, there exists a unique vertex $x \in P$ such that $v \in N_G[x]$. Moreover, if $v \neq x$, then $c(v) = d(x)$.*

Proof. For each $x \in P$, if $d(x) \neq c(x)$, then give the color $r+1$ to all the vertices v in $N_G(x)$ with $c(v) = d(x)$ and then assign $d(x)$ to x. Since $\mathcal{D}_G(P,3) = \emptyset$, no two vertices receiving the color $r+1$ are adjacent. Hence this gives a proper $(r+1)$-coloring f of G. By the construction, f satisfies both (1) and (2). □