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# Unreliable point facility location problems on networks

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ABSTRACT

In this paper we study facility location problems on graphs under the most common optimization criteria, such as, median, center and centdian, but we incorporate in the objective function some reliability aspects. Assuming that facilities may become unavailable with a certain probability, the problem consists of locating facilities minimizing the overall or the maximum expected service cost in the long run, or a convex combination of the two. We show that the $k$-facility problem on general networks is NP-hard. Then, we provide efficient algorithms for these problems for the cases of $k = 1, 2$, both on general networks and on trees. We also explain how our methodology extends to handle a more general class of unreliable point facility location problems related to the ordered median objective function.



## 1. Introduction

In the last years, the literature on facility location has grown a lot, one of the reasons being its close relationship with logistics, distribution systems and communication networks [2,6,15]. This has given rise to the incorporation into location models of very interesting new issues. Among the many that can be found in the literature, in this paper we focus on reliability aspects related to the possibility of disruption of a facility. There are different sources of uncertainty that may cause the disruption of a facility, giving rise to situations where some facilities become temporary unavailable to provide the service to the customers. Some examples are system or connection failures, natural disasters, terrorist attacks, labor strikes, etc. As a result, network systems incur in extra (transportation or communication) costs, since customers originally served by the closest facilities must be redirected to more distant ones (see, e.g., [20]). This has motivated an alternative approach to the "customer-to-closest facility" cost that consists of locating facilities that minimize the total expected service cost in the long run, assuming that failures are accidental, and their probabilities can be estimated in advance [3].

One of the assumptions made in [3] is that the probability of disruption is a function of the facility design and is not dependent on the facility location. Of course, this assumption is valid in some cases and in particular when probabilities of failure are equal, such as, for example, in [37]. On the other hand, it is also possible that the disruption depends on the location of the facility, i.e., it depends on the reliability of the network where it is located and on the possibility that some parts of the network may become unavailable due to structural reasons. In a typical location problem of a service point in a city network, this is possible, for example, when the disruption is related to congestion reasons or labor strikes that usually affect more the service points located in some particular parts of the network. An additional application of our problem is related to the context of network reliability. Actually, evaluation of system reliability is one of the fundamental issues in the design of communication networks, and several indices are presented in the literature for the evaluation of the performance

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**Table 1**
Summary of the results for unreliable facility location problems on networks. Function $\lambda_s(n)$ is the maximum length of a Davenport–Schinzel sequence of order $s$ on $n$ symbols.

| | 1-Facility | | 2-Facility | |
|---|---|---|---|---|
| | Graphs | Trees | Graphs | Trees |
| Median | $O(mn \log n)$ | $O(n)$ | $O(m^2n^3)$ | $O(n^3)$ |
| Center | $O(m\lambda_3(n) \log n)$ | $O(n\lambda_3(n) \log n)$ | $O(m^2n^{4+\varepsilon})$ | $O(n^{5+\varepsilon})$ |
| Centdian | $O(m\lambda_3(n) \log n)$ | $O(n\lambda_3(n) \log n)$ | $O(m^2n^{4+\varepsilon})$ | $O(n^{5+\varepsilon})$ |

of a network system (see, e.g., [22] and the references therein). In a communication network a node acts as server/switch and the failure may derive from a malfunction of the node itself; the same may happen for edges. Basing on the seminal paper by [1], the failure of a node implies the failure of all links incident to it. Therefore, when considering the problem of locating switching centers in a network, the probability of disruption of the facility strictly depends on the probability of failure of the points in the network. This is exactly the probability framework analyzed in this paper. Probabilistic analysis and estimation of failures in power transmission networks are well studied in the literature (see [8]).

In our problem we assume that independent failure probabilities are associated to the vertices of a network and that the failure probabilities of the points in the interior of an edge are induced by the failure probabilities of the end vertices of the edge. For the ease of presentation and without loss of generality, we will assume that the probability of failure of a point in an edge $(u, v)$ is the linear interpolation of the (known) probabilities of failure of the vertices $u$ and $v$. Therefore, the closer a point in an edge to a vertex, the greater the influence of such a vertex on the failure probability of the point. The reader is referred to Section 6 for an explanation of how this approach extends to be applied to general types of probability functions.

It is clear that these probabilities are location dependent, implying that some of the results that were valid in [3] do not longer hold, and thus making our model framework new and challenging. The reader may note that assuming that probabilities are location dependent has been already considered by some authors in the field of discrete location by simply considering different probabilities associated to each facility [4,36,38]. Nevertheless, the related location problem on networks has not been addressed yet in the literature.

The goal of this paper is to analyze reliability issues of location problems under this new failure probability model. To this purpose, we analyze 1- and 2-facility location problems on networks with the most commonly used objective functions: minimizing the overall sum (*median* criterion), the maximum (*center* criterion) expected service cost in the long run, and a convex combination of the sum and maximum (*centdian* criterion). We adopt tools of different types.

On the one hand, we rely on discretization. Since the seminal paper by Hakimi [10], much of the work related to location problems on networks has been devoted to identify a finite set of points such that some optimal solutions of the problem belong to it. This set, called *Finite Dominating Set* (FDS), is very useful in order to restrict the number of possible candidate points to be optimal solutions. This sort of discretization strategy has given very good results and it is at the basis of our approach to solve the problems considered in this paper. An overview of the literature involving characterizations of FDS shows a lot of papers that succeeded in finding such kind of sets. The reader is referred to [12] and to [25] and the references therein as literature sources on this subject (see, also [5,14,17,24,31,33]). More recent references dealing with other multifacility location problems on networks are [13,16]. The former derives a FDS for the $k$-median problem with positive and negative weights; the latter reference solves the 2-facility case for different equity measures.

On the other hand, we shall make an extensive use of arrangements of planar and three dimensional curve patches and of some other tools borrowed from Computational Geometry that will allow us to state the complexity of our algorithms [23,34,35].

In this paper we develop efficient solution algorithms to tackle 1- and 2-facility problems for all the objective functions listed above. There are reasons for distinguishing between the 1- and 2-facility cases. In the 1-facility problem one is assuming that users are willing to try to get the service once before balking, whereas in the 2-facility case, customers will try twice (more than one) before balking which makes a difference in the analysis. Since it is reasonable to assume that in real-life situations a customer does not try more than a very limited number of times, we restrict ourselves to consider the case of two attempts to receive service before giving up. Even if the main interest is for the study of 1- and 2-facility cases, our analysis extends further to any fixed number $k$. It is important to point out that there is a substantial difference between our problems and standard location problems where multifacility issues have a completely different interpretation. In fact, in the classical $k$-facility location problems $k$ points are located but each client is served by only one of them. In our problem, for each client all the $k$ located facilities are different possibilities to be served: following the ordering of the distances, the client will be served first by its closest (operating) facility, then by its second closest, and so on; if all the facilities fail, a penalty cost must be paid for redirecting the client to a backup facility.

We consider the case of general networks, as well as, tree networks. On general networks we observe that the $k$-facility versions of the problems considered in this paper are NP-Hard since they contain the $k$-median, $k$-center and $k$-centdian problem, respectively (see, [18,19]). Our final complexity results for $k = 1, 2$ on general networks and on trees are summarized in Table 1.

The paper is organized as follows. Section 2 presents the notation and the definition of the problems analyzed in the paper. It also contains an interesting counterexample showing that vertex optimality is not ensured in this type of reliability

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