



Note

A note on S -packing colorings of lattices

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ABSTRACT

Let $a_1, a_2, \dots, a_k$ be positive integers. An $(a_1, a_2, \dots, a_k)$ -packing coloring of a graph G is a mapping from $V(G)$ to $\{1, 2, \dots, k\}$ such that vertices with color i have pairwise distance greater than a_i . In this paper, we study $(a_1, a_2, \dots, a_k)$ -packing colorings of several lattices including the infinite square, triangular, and hexagonal lattices. For k small, we determine all a_i such that these graphs have packing colorings. We also give some exact values and asymptotic bounds.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph. A set $X \subseteq V(G)$ is called an i -packing of G if vertices of X have pairwise distance greater than i . Let $a_1, \dots, a_k$ be positive integers. An $(a_1, \dots, a_k)$ -packing coloring of a graph G is a mapping f from $V(G)$ to $\{1, 2, \dots, k\}$ such that vertices with color i have pairwise distance greater than a_i . For brevity, we will call this just an $(a_1, \dots, a_k)$ -coloring. For example, if all the a_i are 1, this represents a normal k -coloring, and if all the a_i are r , this represents a normal k -coloring of the r th power of G . Further, if $S = (a_1, a_2, \dots)$ is a sequence of positive integers, we define the S -packing chromatic number $\chi_S(G)$ of G as the smallest integer k such that G has an $(a_1, \dots, a_k)$ -coloring. If no such coloring of G exists for any positive integer k , then we say $\chi_S(G) = \infty$.

The concept of S -packing chromatic number was mentioned in [8] and formally introduced and studied in [9]. There the focus was on the S -packing chromatic number of the infinite path P_∞ and the computational complexity of (a_1, a_2, a_3) -colorability. A special case of this, called the packing chromatic number and written $\chi_\rho(G)$, is the case when $S = (1, 2, 3, \dots)$. There have been several papers on the packing chromatic number, especially on the values for lattices and other infinite graphs [1–3, 6–8, 12–14], as well as on the computational complexity of the parameter [5, 8].

In this paper, we consider $\chi_S(G)$ of several lattices. The Cartesian product of graphs G and H , written $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where (u, v) and (u', v') are adjacent if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. The two-way infinite path, written P_∞ , is the graph with the integer set $\mathbb{Z}$ as the vertex set, such that two vertices are adjacent if and only if they correspond to consecutive integers. The infinite square lattice, written $\mathbb{Z}^2$, is $P_\infty \square P_\infty$. This graph has been the subject of several papers. The original [8] showed that $9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$; the lower bound was later improved to 10 by Fiala et al. [6] and then to 12 by Ekstein et al. [2], while the upper bound was improved by Holub and Soukal [13] to 17.

The infinite triangular lattice, denoted by $\mathcal{T}$, is the graph obtained from $\mathbb{Z}^2$ by adding all edges of the form $\{(i, j), (i + 1, j - 1)\}$. The infinite hexagonal lattice, denoted by $\mathcal{H}$, is the 3-regular infinite plane graph where every face is a hexagon. Fiala et al. [6] showed that $\chi_\rho(\mathcal{H}) = 7$, while Finbow and Rall [7] showed that $\chi_\rho(\mathcal{T}) = \infty$.

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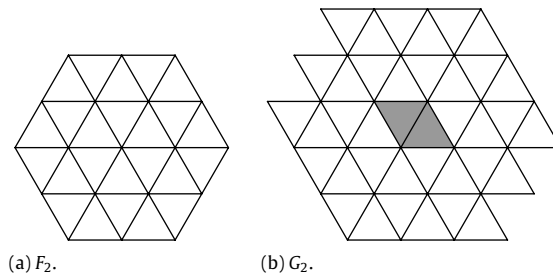


Fig. 1. Subgraphs containing only one vertex of a 7- or 9-packing in $\mathcal{H}$ (shown in dual).

In this paper, we investigate $\chi_S(G)$ of the infinite one-row, square, two-row, triangular, and hexagonal lattices. We determine the cases where χ_S is small for these graphs. We also investigate the parameter for some special sequences, and give some exact values and asymptotic bounds.

2. Some tools

2.1. Definitions

Throughout the paper, every sequence S is a nondecreasing sequence of positive integers, and a_i denotes the i th term of S . We define a sequence $(a_1, \dots, a_k)$ as a *minimal packing chromatic sequence* (MPCS) for a graph G if G has an $(a_1, \dots, a_k)$ -coloring but does not have such a coloring if any of the a_i are increased. Note that an MPCS might not exist: for example, a star has a $(1, a_2)$ -coloring for all a_2 . But MPCS exist for the lattices we consider.

A natural way to construct suitable colorings is to take an existing good coloring and partition one of the classes into several classes. We call such a process *splitting*.

2.2. Density

We will use X_i to denote an i -packing in the graph. Upper bounds on the size of X_i are useful in establishing lower bounds on $\chi_S(G)$. Density was formally introduced by Fiala et al. [6].

Definition 1 ([6]). Let G be a graph with finite maximum degree. Then the *density* of a set of vertices $X \subset V(G)$ is

$$D_G(X) = \limsup_{r \rightarrow \infty} \max_{v \in V} \left\{ \frac{|X \cap N_r[v]|}{|N_r[v]|} \right\}$$

where $N_r[v]$ is the set of vertices within distance r of v .

Lemma 1 ([6]). If an $(a_1, a_2, \dots, a_k)$ -coloring of G has classes $X_{a_1}, X_{a_2}, \dots, X_{a_k}$, then

$$\sum_{i=1}^k D_G(X_{a_i}) \geq 1.$$

Lemma 2 ([6]). In the infinite square lattice $\mathbb{Z}^2$,

- (a) $D_{\mathbb{Z}^2}(X_k)$ is at most $2/(k+1)^2$ if k is odd, and at most $2/(k^2+2k+2)$ if k is even;
- (b) $D_{\mathbb{Z}^2}(X_1 \cup X_2) \leq \frac{5}{8}$.

We will also need the bounds for the hexagonal lattice. The following lemma uses some ideas from [10].

Lemma 3. In the infinite hexagonal lattice $\mathcal{H}$,

- (a) $D_{\mathcal{H}}(X_k) \leq 1/6m^2$ if $k = 4m - 1$;
- (b) $D_{\mathcal{H}}(X_k) \leq 1/(6m^2 + 6m + 2)$ if $k = 4m + 1$; and
- (c) $D_{\mathcal{H}}(X_k) \leq 2/(3m^2 + 3m + 2)$ if $k = 2m$.

Proof. (a) We decompose $\mathcal{H}$ into isomorphic subgraphs F_m whose vertices are distance at most $4m - 1$ apart in $\mathcal{H}$. When $m = 1$, let F_m be a 6-cycle. When $m > 1$, let F_m include all the vertices within distance $2m - 2$ of a 6-cycle (including the 6-cycle itself). It is easy to see that F_m has order $6m^2$ and diameter $2(2m - 2) + 3 = 4m - 1$, and so contains at most one vertex of a $(4m - 1)$ -packing. The case of F_2 is illustrated in Fig. 1(a), where we use the dual graph $\mathcal{T}$ to represent $\mathcal{H}$.

(b) We again decompose $\mathcal{H}$ into isomorphic subgraphs. When $m = 0$, let G_0 be a pair of adjacent vertices in $\mathcal{H}$. When $m > 0$, let G_m include all the vertices within distance $2m$ of a pair of adjacent vertices (including the pair itself). The subgraph has order $2((2m + 1)^2 - 2(m + 1)m/2) = 6m^2 + 6m + 2$ and diameter $4m + 1$, and so contains at most one vertex of a $(4m + 1)$ -packing. The case of G_2 is illustrated in Fig. 1(b).

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