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More on the one-dimensional sliding-coin puzzle

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ABSTRACT

Consider a line of n nickels and n pennies with all nickels arranged to the left of all pennies, where $n \geq 3$. The puzzle asks the player to rearrange the coins such that nickels and pennies alternate in the line. In each move, the player is allowed to slide k adjacent coins to new positions without rotating. We first prove that for any integer $k \geq 2$ it takes at least n moves to achieve the goal. A well-known optimal solution for the case $k = 2$ matches the lower bound. We also give optimal solutions for the case $k = 3$.

1. Introduction

1.1. Preliminary: an old puzzle for $k = 2$

The sliding-coin puzzle is a solitaire game, which has numerous variants. The puzzle is defined with an initial configuration and a final configuration, and is associated with a moving rule. Consider the configuration on the left of Fig. 1, with three nickels to the left of three pennies. In this paper, we use a white stone and a black stone to denote a nickel and a penny, respectively. The player is allowed to slide a pair of adjacent coins to unoccupied positions in each move. The goal is to reach one of the configurations on the right of Fig. 1, such that black stones and white stones alternate in the line. The final configuration may not completely stay within the initial positions along the line, i.e. it may shift to the left or right as a whole.

The above puzzle can be solved in three moves, as shown in Fig. 2. We write $(i \rightarrow j)$ to denote a move such that two adjacent coins at positions i and $i + 1$ are moved to positions j and $j + 1$, respectively. Note that the player cannot rotate the pair of coins while sliding.

Since all the coins are arranged in the line, we call this puzzle the *one-dimensional sliding-coin puzzle*. In fact, this is an old puzzle. It has been illustrated as "Sheep and goats" [7], and as "empty and filled wine glasses" [4]. There is a very nice solution for $k = 2$ and general n shown in Kordemsky's famous puzzle book[6]. However, there was no clue about the optimality. We show that the solution is optimal.

Actually, there are many sliding-coin puzzles whose coin configurations are arranged on a plane. For two-dimensional sliding-coin puzzles, we refer to [3,5,2,1,8].

Unlike two-dimensional sliding-coin puzzles, the one-dimensional puzzle can be easily extended by adding the same number of nickels and pennies to the configuration or allowing the player to slide more coins in each move. These generalizations form a family of puzzles, and thus we can investigate the complexity of achieving an optimal solution. In this paper, we show a lower bound and some properties of optimal solutions, and explicitly solve the puzzle family optimally for

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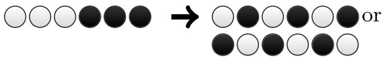

Fig. 1. Rearrange the configuration by sliding a pair of adjacent coins in each move.

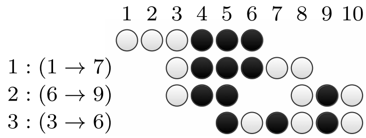

Fig. 2. A three-move solution to the puzzle for $n = 3$.

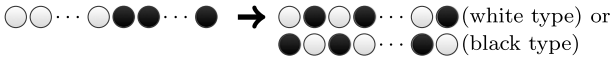

Fig. 3. Rearrange the left configuration into the right configuration by sliding k adjacent stones in each move.

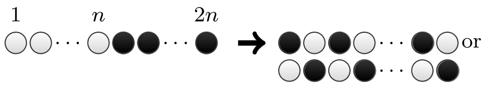

Fig. 4. The initial configuration has one pair of adjacent positions with different colors. The final configuration has $2n - 1$ pairs of adjacent positions with different colors.

$k = 3$. To the best of our knowledge, the one-dimensional sliding-coin puzzle for $k = 3$ has not been studied before. With similar ideas, we can solve optimally the puzzles for $k = 4$ and 5. For larger k, we leave the solution as an open question.

1.2. Definition and notions

For convenience, we give the following definitions.

Definition 1 (*($i \to j$)-Move*)**.** An $(i \to j)$-move slides k adjacent stones, without rotating, from source positions $i, i + 1, \ldots, i + k - 1$ to destination positions $j, j + 1, \ldots, j + k - 1$, respectively.

The initial configuration is as shown on the left of the arrow in Fig. 3, with n white stones to the left of n black stones for $n \geq 3$. The final configuration is called *white type* if the first stone is white; otherwise, it is *black type*. We want to rearrange the initial configuration into one of the final configurations via a sequence of valid moves.

Definition 2 (*(n, k)-Game*)**.** Given integers $k \geq 2$ and $n \geq 3$, we call the corresponding sliding coin puzzle the (n, k)-game.

For example, the puzzle shown in Fig. 1 is a (3, 2)-game. We say that a stone becomes *stationary* in a solution if it is not slid anymore in the rest of the moves, i.e. it has reached the correct position as in the final configuration. For example, in Fig. 2, the stone at position 5 and the stone at position 8 in row 1 are both stationary. We say a stone is stationary in a solution if it is stationary from the very beginning.

We denote a solution as *optimal* if there is no other solution using fewer moves. It is common for a puzzle to have more than one optimal solution. A *k-block* indicates k adjacent consecutive positions in a configuration.

The rest of the paper is organized as follows. Section 2 studies the lower bound and some general properties of optimal solutions. For completeness, in Section 3, we show a recursive algorithm for the $(n, 2)$-game which generates optimal solutions. Section 4 gives two algorithms for the $(n, 3)$-game, one for odd n and the other for even n, since the optimal solutions of the two cases are very different. Section 5 concludes with some open problems.

2. Lower bound and properties

We study some properties about optimal solutions which are independent of k. These will help in searching for the solutions. We first show the minimum number of moves required to solve a puzzle by counting adjacent positions with white and black stones, i.e. ○● or ●○.

Theorem 2.1. *For integers $n \geq 3$ and $k \geq 2$, it takes at least n moves to solve the (n, k)-game.*

Proof. We prove the theorem by contradiction. Suppose that the game can be solved in $n-1$ moves. Observe that, in Fig. 4, in the initial configuration there is only one pair of positions with different types of stone, i.e. the nth stone being white and the