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Cooperative assignment games with the inverse Monge property

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ABSTRACT

We study inverse-Monge assignment games, namely cooperative assignment games in which the assignment matrix satisfies the inverse-Monge property. For square inverse-Monge assignment games, we describe their cores and we obtain a closed formula for the buyers-optimal and the sellers-optimal core allocations. We also apply the above results to solve the non-square case.

1. Introduction

The Monge property of a matrix was named by Hoffman [4] in recognition of the work of the 18th-century French mathematician Gaspard Monge, who used the property to solve a soil-transport problem. The property has been subsequently applied to a variety of areas — for specific references, applications and properties see the surveys in Burkard [2] or Burkard et al. [3].

Here, we show that by using Monge matrices, the analysis of cooperative assignment games is simplified thanks to the Monge conditions and properties. Our main interest lies in describing the core and two specific allocations — the buyers-optimal and the sellers-optimal core allocations, when dealing with inverse-Monge assignment games.

The optimal (linear sum) assignment problem is that of finding an optimal matching, given a matrix that collects the potential profit of each pair of agents of opposite sectors. Examples include the placement of workers in jobs, students in colleges, or physicians in hospitals. Once an optimal matching has been made, the question arises as how to share the output between partners. This question was first considered in Shapley and Shubik [8]. They associate each assignment problem with a cooperative game, or game in coalitional form. In the assignment game, the worth of each coalition of agents is taken as the maximum profit they can attain.

The main solution concept in cooperative games is the core. The core of a game consists of allocations of the optimal profit (the worth of the grand coalition) in such a way that no subcoalition can further improve upon it. Thus, if the parties agree to share the profit of cooperation by means of a core allocation, no coalition has any incentive to depart from the grand coalition and act on its own. Shapley and Shubik prove that the core of the assignment game is a nonempty polyhedral convex set and that this coincides with the set of solutions of the dual linear program related to the linear sum optimal assignment problem.

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In this paper we study assignment games in which the matrix satisfies the inverse Monge property, called inverse-Monge assignment games. This property is equivalent to the supermodularity of the matrix, interpreted as a function on the product of the set of indices with the usual order.

We show that for square inverse-Monge assignment games, the central tridiagonal band of the matrix, that is the main diagonal, the upper diagonal and the lower diagonal, is sufficient to determine the core. As a result, and unlike the general case, not all inequalities are necessary to describe the core explicitly, and in this case the buyer–seller exact representative of the matrix [6] can be computed by a closed formula. Two extreme points in the core, the buyers-optimal and the sellers-optimal core allocations, are computed with the aid of the previous representation.

The paper is organized as follows. In Section 2 we describe the assignment game and the pertinent results. In Section 3, inverse-Monge assignment games are defined and we describe the core of a square inverse-Monge assignment game. In Section 4 we give an explicit formula to compute the buyers-optimal and the sellers-optimal core allocations. In Section 5, the non-square case is considered. We conclude in Section 6 with some remarks.

2. The assignment game

An assignment problem (M, M', A) is defined to be a nonempty finite set M of agents, usually named buyers, a nonempty finite set M' of another type of agents, usually named sellers, and a nonnegative matrix $A = \left(a_{ij}\right)_{(i,j)\in M\times M'}$. Entry a_{ij} represents the profit obtained by the mixed-pair $(i, j) \in M \times M'$ if they trade. Let us assume there are $|M| = m$ buyers and $\left|M'\right| = m'$ sellers. If $m = m'$, the assignment problem is said to be square. Let us denote by $\mathrm{M}^{+}_{m\times m'}$ the set of nonnegative matrices with m rows and m' columns.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M'_0 \subseteq M'$, such that $|M_0| = \left|M'_0\right| = \min\left\{|M|, \left|M'\right|\right\}$. We write $(i, j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. The set of all matchings is denoted by $\mathcal{M}\left(M, M'\right)$. A buyer $i \in M$ is unmatched by μ if there is no $j \in M'$ such that $(i, j) \in \mu$. Similarly, $j \in M'$ is unmatched by μ if there is no $i \in M$ such that $(i, j) \in \mu$.

A *matching* $\mu \in \mathcal{M}\left(M, M'\right)$ is *optimal* for the assignment problem $\left(M, M', A\right)$ if for all $\mu' \in \mathcal{M}\left(M, M'\right)$ we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathcal{M}^*_A\left(M, M'\right)$.

Shapley and Shubik [8] associate any assignment problem with a game in coalitional form (*assignment game*) with player set $N = M \cup M'$ and characteristic function w_A defined by A in the following way: for $S \subseteq M$ and $T \subseteq M'$,

$$w_A(S \cup T) = \max_{\mu\in\mathcal{M}(S,T)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\}, \tag{1}$$

where $\mathcal{M}(S, T)$ is the set of matchings from S to T and $w_A(S \cup T) = 0$ if $\mathcal{M}(S, T) = \varnothing$.

The *core* of the assignment game,[1]

$$C(w_A) = \left\{ (x, y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \middle| \begin{array}{l} x(S) + y(T) \geq w_A(S \cup T), \\ \text{for all } S \subseteq M \text{ and } T \subseteq M', \text{ and} \\ x(M) + y(M') = w_A\left(M \cup M'\right) \end{array} \right\},$$

is always nonempty and, if $\mu \in \mathcal{M}^*_A(M, M')$ is an arbitrary optimal matching, the core is the set of nonnegative payoff vectors $(u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ such that

$$u_i + v_j \geq a_{ij} \quad \text{for all } (i, j) \in M \times M', \tag{2}$$

$$u_i + v_j = a_{ij} \quad \text{for all } (i, j) \in \mu, \tag{3}$$

and the payoff to unmatched agents by μ is null. This coincides (see Shapley and Shubik [8]) with the set of solutions of the dual of the linear program related to the linear sum assignment problem.

Among the core allocations of an assignment game, there are two specific extreme core points: the *buyers-optimal core allocation* $(\overline{u}^A, \underline{v}^A)$ where each buyer attains her maximum core payoff and each seller his minimum, and the *sellers-optimal core allocation* $(\underline{u}^A, \overline{v}^A)$ where each seller attains his maximum core payoff and each buyer her minimum. From Roth and Sotomayor [7] we know that the maximum payoff of an agent is his/her marginal contribution, and that this can be attained for all agents on the same side at the same core allocation. For any assignment game $(M \cup M', w_A)$, we have

$$\begin{aligned} \overline{u}^A_i &= w_A(M \cup M') - w_A(M \cup M' \setminus \{i\}) \quad \text{for all } i \in M, \quad \text{and} \\ \overline{v}^A_j &= w_A(M \cup M') - w_A(M \cup M' \setminus \{j\}) \quad \text{for all } j \in M'. \end{aligned} \tag{4}$$

[1] For any vector $z \in \mathbb{R}^N$, with $N = \{1, \ldots, n\}$ and any coalition $R \subseteq N$ we denote by $z(R) = \sum_{i\in R} z_i$. As usual, the sum over the empty set is zero.