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Deterministic walks with choice

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ABSTRACT

This paper studies deterministic movement over toroidal grids (and more generally, Cayley graphs on groups), integrating local information, bounded memory and choice at individual nodes. The research is motivated by recent work on deterministic random walks, and applications in multi-agent systems. Several results regarding passing tokens through toroidal grids are discussed, as well as some open questions.

1. Introduction

This paper studies the movement of an item across a Cayley graph, wherein routing is accomplished through the utilization of choice functions and bounded memory at nodes. Rotor-router walks have been of particular interest recently. Therein, the walker is routed through each node via some cyclic or other order (see for instance [5–7,9,12,14,18,20]). Here, we consider deterministic routing based not only on recent exits, but recent entries, as well. This aligns with recent work on locally fair exploration of graphs with undirected edges (see [8]), but for a given choice process, the dynamics here are entirely deterministic. Of interest is determination of the sets of points visited infinitely often for choice processes (homogeneous across the graph).

The following example is illustrative of the contrast between simple rotor router walks and deterministic walks with choice.

Example 1 (*Deterministic Walks with Choice and Rotor Router Walks*)**.** In a simple rotor router model, upon entrance to a node, a walker is routed via a router placed at the node, which rotates upon each entrance (see for instance [14] and the references therein). In Fig. 1, the walker enters a section of a grid at the south-west with routers oriented as shown on the left. Upon entering the node with current orientation east (E), the local router rotates 90° counter-clockwise, and the walker is sent north (N). It is then sent east, east, west, north, east and finally north, in succession, as depicted on the right, before exiting at the north-east corner.

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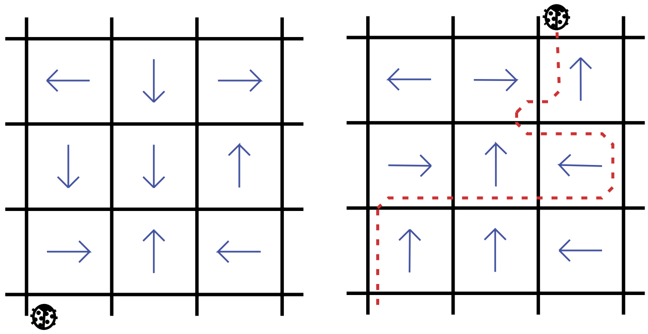

Fig. 1. A rotor router walk.

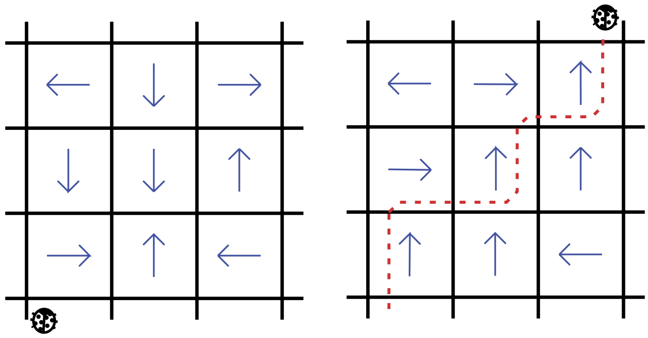

Fig. 2. A deterministic walk with choice.

In contrast, a walk with choice (in this case with "memory" two) is depicted in Fig. 2. Therein, the directions shown on the left are the most recent outgoing directions from the node; the walker again enters the grid section at the south-west. Having entered from the south, the walker is then sent in one of the two remaining (recently underrepresented) directions for that node—north and west. In this case the walker is routed north (denoted $C(\{\mathrm{N}, \mathrm{W}\}) = \mathrm{N}$). At the next node, the last outgoing direction was south and the incoming direction is also south, so a decision arises to choose between north, east and west, assuming the decision made is east (i.e. $C(\{\mathrm{N}, \mathrm{E}, \mathrm{W}\}) = \mathrm{E}$), the walker is sent to the middle node. If, in addition to the aforementioned choices, we have that $C(\{\mathrm{N}, \mathrm{E}\}) = \mathrm{N}$ and $C(\{\mathrm{N}, \mathrm{S}\}) = \mathrm{N}$, the walker will proceed as on the right, eventually exiting as before at the north-east corner. Note that, as mentioned, we are assuming homogeneity of choice (both at individual nodes and across the graph), so that, for instance, when a selection is made from the set $\{\mathrm{N}, \mathrm{W}\}$, it is always N, regardless of time and location.

Remark. There has been considerable recent work focused on walks wherein consideration is placed on edges covered. In [3], the authors consider an *edge process* wherein, upon entering a node, a walker is sent along a particular edge selected from among the unpassed edges incident to the node. Once all incident edges are passed the walker is sent uniformly at random to one of the neighbours (see also the *greedy random walk* in [23,8,24,4] for overall general discussion of related models).

Note that all decisions at nodes for deterministic walks with choice are made without the need to query neighbours or to carry global information on visits to nodes (common in some random walks with memory; see for instance [2,10]). For a survey of results on self-avoiding random walks, see [25].

Consideration of avoidance of recent entry sources (as well as exits) is suggested by results on improved mixing times for non-backtracking random walks (see for instance [1,21,13]). The walk with choice suggested in Example 1 is an example of a non-backtracking deterministic walk. □

1.1. Preliminaries and notation

Suppose $k \geq 0$, and consider a group G with finite (generating) subset $\mathcal{S} = \{s_1, s_2, \ldots, s_w\}$ (closed under inverses), identity element e, and Cayley graph $\mathrm{Cay}(G, \mathcal{S})$ whose vertices are the elements of G and whose edges are ordered pairs (g, gs) for $g \in G$ and $s \in \mathcal{S}$.

Now, consider a choice function C on the non-empty subsets of $\mathcal{S}$ (i.e. C is a function from $\mathcal{P}(\mathcal{S}) \setminus \emptyset$ to $\mathcal{S}$, where $\mathcal{P}(\mathcal{S})$ is the power set of $\mathcal{S}$), and for $Y_0 \in G$, set $I_0 = e$ and $O_0 = C(\mathcal{S})$. We define further values of the sequence $\{\mathcal{Y}_i\}_{i \geq 0} = \{(Y_i, I_i, O_i)\}_{i \geq 0}$,