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Minimum degree, independence number and pseudo [2, b]-factors in graphs

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ABSTRACT

A pseudo [2, b]-factor of a graph G is a spanning subgraph in which each component C on at least three vertices verifies $2 \leq d_C(x) \leq b$, for every vertex x in C. Given an integer $b \geq 4$, we show that a graph G with minimum degree δ, independence number $\alpha > \frac{b(\delta-1)}{2}$ and without isolated vertices possesses a pseudo [2, b]-factor with at most $\alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ components that are edges or vertices. This bound is sharp.

1. Introduction

Throughout this paper, graphs are assumed to be finite and simple. For unexplained concepts and notations, the reader could refer to [1].

Given a graph G, we let $V(G)$ be its vertex set, $E(G)$ its edge set and n its order. The neighborhood of a vertex x in G is denoted by $N_G(x)$ and defined to be the set of vertices of G adjacent to x; the cardinality of this set is called the degree of x in G. For convenience, we denote by $d(x)$ the degree of a vertex x in G, by δ the minimum degree of G and by α its independence number. However, if H is a subgraph of G then we write $d_H(x)$, δ_H, and $\alpha(H)$ for the degree of x in H, the minimum degree, and the independence number of H respectively. We denote by $d_G(x, y)$ the distance between x and y in the graph G.

A factor of G is a spanning subgraph of G, that is a subgraph obtained by edge deletions only. If S is the set of deleted edges, then this subgraph is denoted $G - S$. If H is a subgraph of G, then $G - H$ stands for the subgraph induced by $V(G) - V(H)$ in G. By starting with a disjoint union of two graphs G_1 and G_2 and adding edges joining every vertex of G_1 to every vertex of G_2, we obtain the join of G_1 and G_2, denoted $G_1 + G_2$. For a positive integer p, the graph pG consists of p vertex-disjoint copies of G. In all that follows, we use disjoint to stand for vertex-disjoint.

In [2], we defined a pseudo 2-factor of a graph G to be a factor each component of which is a cycle, an edge or a vertex. It can also be seen as a graph partition by a family of vertices, edges and cycles. Graph partition problems have been studied in lots of papers. They consist of partitioning the vertex set of G by disjoint subgraphs chosen to have some specific properties. In [3], Enomoto listed a variety of results dealing with partitions into paths and cycles. The emphasis is generally on the existence of a given partition. In our study of pseudo-factors, however, we take interest in the number of components that are edges or vertices in a pseudo-factor of G. In [2], we proved that every graph with minimum degree $\delta \geq 1$ and independence number $\alpha \geq \delta$ possesses a pseudo 2-factor with at most $\alpha - \delta + 1$ edges or vertices and that this bound is best possible. Motivated by the desire to know what happens in general cases, we define a *pseudo* [a, b]*-factor* (where a and b are two integers such that $b \geq a \geq 2$) as a factor of G in which each component C on at least three vertices verifies

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$a \le d_C(x) \le b$, for every $x \in C$. Clearly, a pseudo $[a, b]$-factor with no component that is an edge or a vertex is nothing but an $[a, b]$-factor. Surveys on factors and specifically $[a, b]$-factors and connected factors can be found in [4,5]. In the present work, we study pseudo $[2, b]$-factors, we consider the case $b \ge 4$ and obtain an upper bound (in function of δ, α and b) for the number of components that are edges or vertices in a pseudo $[2, b]$-factor of G which minimizes the number of such components. Note that, from a result by Kouider and Lonc [6], we deduce that if $\alpha \le \frac{b(\delta-1)}{2}$ then G has a $[2, b]$-factor. Laying down the condition $\alpha > \frac{b(\delta-1)}{2}$, the main result of this paper reads as follows:

Theorem 1. *Let b be an integer such that $b \ge 4$ and G a graph of minimum degree $\delta \ge 1$ and independence number α with $\alpha > \frac{b(\delta-1)}{2}$. Then G possesses a pseudo $[2, b]$-factor with at most $\alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ components that are edges or vertices.*

The bound given in Theorem 1 is best possible. Indeed, let b be an integer such that $b \ge 4$ and let H be a graph with nonempty set of vertices and empty set of edges. The graph $G = H + pK_2$, where $p > \frac{b}{2}|H|$, has minimum degree $\delta = |H| + 1$ and independence number $\alpha = p$. We can easily verify that G possesses a pseudo $[2, b]$-factor with $\alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ edges and we cannot do better. Also, a simple example reaching the bound of Theorem 1, is a graph G obtained by taking a graph H on n vertices in which every vertex is of degree between 2 and b ($b \ge 4$), then taking n additional independent vertices and joining exactly one isolated vertex to exactly one vertex of H. The graph G has minimum degree $\delta = 1$, independence number $\alpha = n$ and can be partitioned into one component that is H and $n = \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ vertices (or simply n edges) and we cannot do better.

Combining Theorem 1 with the results of [2] and [6], we obtain

Corollary 1. *Let $b \ge 2$ be an integer such that $b \ne 3$. Let G be a graph of minimum degree δ and independence number α and without isolated vertices. Then G possesses a pseudo $[2, b]$-factor with at most $\max(0, \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor)$ edges or vertices.*

2. Independence number, minimum degree and pseudo [2, *b*]-factors

First of all, we put aside the case $\delta = 1$ for which we know that we have in G a pseudo $[2, b]$-factor with at most α edges or vertices. Indeed, if we regard a cycle as a component each vertex of which is of degree between 2 and b, then we know that any graph G can be covered by at most α cycles, edges or vertices (see for instance [7]). So the bound $\alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ holds for $\delta = 1$.

From now on, we assume that G has minimum degree $\delta \ge 2$. Let F be a subgraph of G such that $2 \le d_F(x) \le b$ for all $x \in V(F)$. For the sake of simplifying the writing, such a subgraph F will be called a $[2, b]$-subgraph of G. Denote by D a smallest component of $G - F$, set $W = G - (D \cup F)$ and choose F in such a manner that:

(a) $\alpha(G - F)$ is as small as possible;

(b) subject to (a), the number of vertices of D is as small as possible;

(c) subject to (a) and (b), the number of vertices in F is as small as possible.

Notice that a subgraph F satisfying the conditions above exists since G contains a cycle (as $\delta \ge 2$). We shall show the following theorem which yields Theorem 1:

Theorem 2. *Let b be an integer such that $b \ge 4$. Let G be a graph of minimum degree $\delta \ge 2$ and independence number α such that $\alpha > \frac{b(\delta-1)}{2}$. Then there exists a pseudo $[2, b]$-factor of G and a $[2, b]$-subgraph F of this pseudo $[2, b]$-factor such that $\alpha(G - F) \le \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$.*

Proof of Theorem 2. Let F be a $[2, b]$-subgraph of G satisfying the conditions (a), (b) and (c). Denote by $u_1, \dots, u_m$ ($m \ge 1$) the neighbors of D on F and by P_{ij} a path with internal vertices in D joining two vertices u_i and u_j with $1 \le i, j \le m$ and $i \ne j$. The proof of Theorem 2 will be divided into several claims. The following one, which will intensively be used, reminds Lemma 1 in [2].

Claim 1. *Let F' be a $[2, b]$-subgraph of G which contains the neighbors of D in F and at least one vertex of D. Setting $W' = G - (F' \cup D)$, we have $\alpha(W') > \alpha(W)$.*

Proof of Claim 1. Set $D' = D - F'$.

(1) If $D' = \emptyset$, then by the choice of F, we have $\alpha(G - F) \le \alpha(G - F')$. But $\alpha(G - F) = \alpha(W) + \alpha(D) \ge \alpha(W) + 1$ and $\alpha(G - F') = \alpha(W')$, so $\alpha(W) < \alpha(W')$.

(2) If $D' \ne \emptyset$, then F' gives a component D' smaller than D, so again by the choice of F, we have $\alpha(W) + \alpha(D) = \alpha(G - F) < \alpha(G - F') = \alpha(W') + \alpha(D')$. But as $\alpha(D') \le \alpha(D)$, we obtain $\alpha(W) < \alpha(W')$. □

In the next claims, we try to learn more about the degrees in F of its vertices.

Claim 2. *For every i, $1 \le i \le m$, we have $N_F(u_i) \cap \{u_1, \dots, u_m\} = \emptyset$.*

Proof of Claim 2. Suppose that for some i, $N_F(u_i) \cap \{u_1, \dots, u_m\} \ne \emptyset$, then there exists a vertex u_j ($1 \le j \le m$ and $j \ne i$) such that $u_iu_j \in E(F)$. Put $e = u_iu_j$, then $(F - e) \cup P_{ij}$ is a $[2, b]$-subgraph. Indeed, none of the vertices of F changes its degree in $(F - e) \cup P_{ij}$ and the internal vertices of P_{ij} are of degree 2. So taking $F' = (F - e) \cup P_{ij}$ in Claim 1 we obtain $\alpha(W) > \alpha(W)$, which is absurd. □